Change of variables

April 2020

Evaluation of a multiple integral by means of iterated integrals can be a complicated process. Both the **integrand** and the **region of integration** can contribute to computational difficulties.

Our goal for this section is to see ways in which changes in coordinates can be used to transform iterated integrals into ones that are relatively simple to compute.

Consider the following theorem.

Theorem of change of variables in double integrals

Let D and D^* be elementary regions in (respectively) the xy-plane and the uv-plane. Suppose $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a coordinate transformation of class C^1 that maps D^* onto D in a one-one fashion. If $\mathbf{f} : D \longrightarrow \mathbb{R}$ is any integrable function and we use the transformation $\mathbf{T}$ to make the substitution $x = x(u, v)$, $y = y(u, v)$, then

$$\int\int_D f(x, y) \; dxdy = \int\int_{D^*} f(x(u, v), y(u, v)) \cdot |\det(\mathbf{J})| \; dudv$$

where J is the Jacobian matrix of the transformation:

$$J = \begin{pmatrix} \dfrac{\partial x(u, v)}{\partial u} & \dfrac{\partial x(u, v)}{\partial v} \\ \dfrac{\partial y(u, v)}{\partial u} & \dfrac{\partial y(u, v)}{\partial v} \end{pmatrix}$$

Example

Compute $\int\int_D \cos(x + 2y) \sin(x - y) \; dxdy$, where D is the triangular region bounded by the lines $y = 0$, $y = x$ and $x + 2y = 8$.

In order to separate the cosine and sine in the integrand let's consider the following transformation: $u = x + 2y$ and $v = x - y$. Solving for x and y in the linear system we get:

$$x = \frac{u + 2v}{3}, \; y = \frac{u - v}{3}$$

So, the application $T(u, v)$ is:

$$T(u, v) = \left(\frac{u + 2v}{3}, \frac{u - v}{3}\right)$$

1

Then the Jacobian matrix J is:

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} T_u & T_v \\ T_u & T_v \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

So, $|\det(J)| = -\dfrac{1}{3}$.

Finally, to identify D^* we see how does the boundary of D transform:

- $y = x \longrightarrow x - y = 0 \longrightarrow \mathbf{v = 0}$

- $x + 2y = 8 \longrightarrow \mathbf{u = 8}$

- $y = 0 \longrightarrow \frac{u-v}{3} = 0 \longrightarrow \mathbf{u = v}$

We have the following sketch of the regions in the uv-plane and xy-plane:

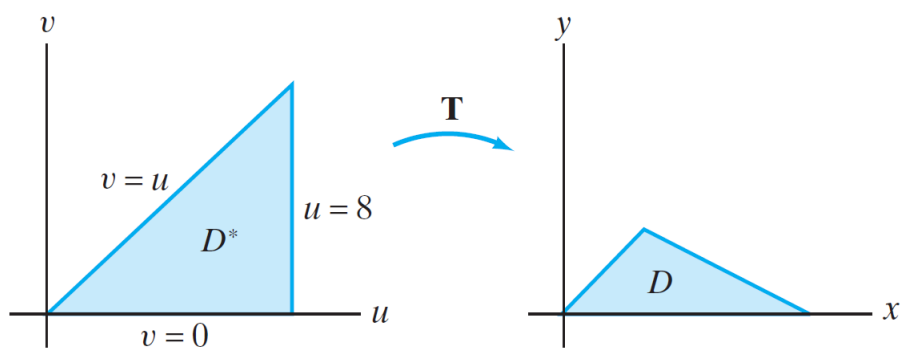Figure 1: Transformation of the regions trough $\mathbf{T}$.

Now we use the theorem to compute the integral:

$$\int\int_D \cos(x+2y)\sin(x-y)\ dxdy = \int\int_{D^*} \cos(u)\sin(v)|det(J)|\ dudv = \int_0^8 \int_0^u \frac{1}{3}\cos(u)\sin(v)\ dvdu$$

$$= \ldots =$$

$$= \frac{1}{3}\int_0^8 \cos(u) - \cos^2(u) = \frac{1}{3}\Big[\sin(u)\Big|_0^8 - \int_0^8 \frac{1}{2}(1-\cos(2u)\ du\Big]$$

$$= \ldots =$$

$$= \frac{1}{3}\Big[\sin 8 - 4 - \frac{1}{4}\sin 16\Big]$$

$\square$

2

OBSERVATION: As you may see in the previous example linear transformations, **T**, transform lines into lines. Also, vertices converts to vertices, and parallelograms transforms to another parallelogram. For example the linear application $\mathbf{T}(u, v) = (u + v, u - v)$ performs the following transformation:
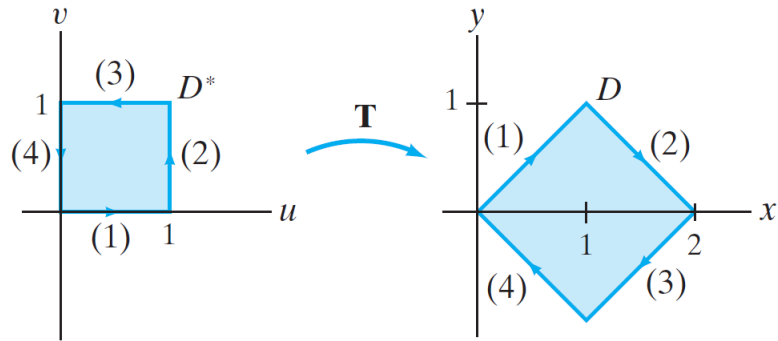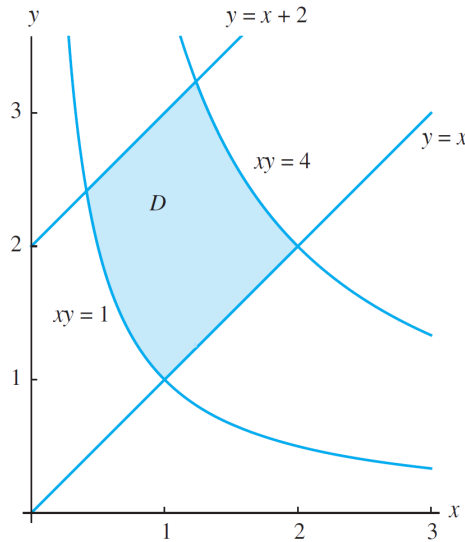

To see this in more detail have a look to proposition 5.1 in the book.

Example

Compute $\int \int_D (x^2 - y^2)e^{xy} \, dxdy$ where D is the region in the first quadrant bounded by the **hyperbolas** $xy = 1$, $xy = 4$, and the **lines** $y = x$, $y = x + 2$.

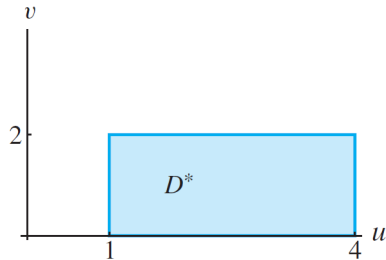A sketch of the region of integration is the following:

Both the integrand and the region present complications for evaluation. We can try to make a change

of variables to simplify the integrand or try a change to simplify the region. Let's perform the former one. Consider the change of variables:

- $u = xy$

- $v = y - x$

Then the region D^* will be a rectangle:

The hyperbolas transforms in vertical lines ($u = 1$, $u = 4$) and the lines transformed in horizontal lines ($v = 0$, $v = 2$).

We need to invert

$$T^{-1}(x, y) = \begin{cases} u = xy \\ v = y - x \end{cases}$$

for computing J, but it will not be necessary. Because determinants properties we have the following relation between the maps $T(u, v)$ and $T^{-1}(x, y)$:

$$\det(J_T) = \frac{1}{\det(J_{T^{-1}})}$$

In our case we have that,

$$J_{T^{-1}} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} y & x \\ -1 & 1 \end{pmatrix}$$

So,

$$\det(J) = \frac{1}{\det(J_{T^{-1}})} = \frac{1}{\begin{vmatrix} y & x \\ -1 & 1 \end{vmatrix}} = \frac{1}{x + y}$$

Since we will be working in the first quadrant (where x and y are both positive), no need to take the absolute value.

At last we are ready to compute:

$$\int\int_D (x^2 - y^2)e^{xy} \, dxdy = \int\int_{D^*} (x^2 - y^2)e^{xy}\frac{1}{x+y} \, dudv = \int_0^2 \int_1^4 \frac{(x-y)(x+y)}{x+y}e^{xy} \, dudv$$

$$= \int_0^2 \int_1^4 -ve^u \, dudv = \dots = 2(e - e^4)$$

$\square$

Example - Polar coordinates

We want to focus on the following change of coordinates:

$$\int\int_D f(x,y,) \, dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta) \, |\det(J)| \, drd\theta$$

Lets find the Jacobian matrix. In general we have that

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Then,

$$J = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \implies \det(J) = r$$

since we take the radius to be positive, no need to take the absolute value.

OBSERVATION: This change of coordinates transform squares in circles, as you may see in the following sketch:
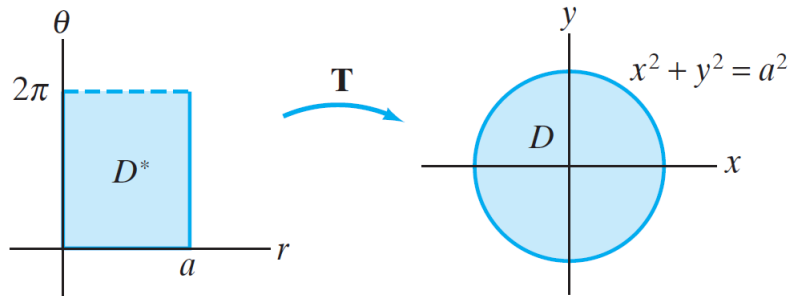

Figure 2: **T** maps the (nonclosed) rectangle D^* to the disk D of radius a.

Example

Compute the area of a circle with double integrals.

Let's see polar coordinates "in action". Let D be the disk of radius a, centred at the origin. Then we have,

$$\text{Area}_\circ = \int\int_D 1 \, dA = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dxdy = \int_0^{2\pi} \int_0^{a} r \, drd\theta = \ldots = \pi a^2$$