Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 16

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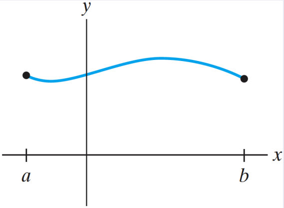
Outline

Definite Integral of a Function of One Variable

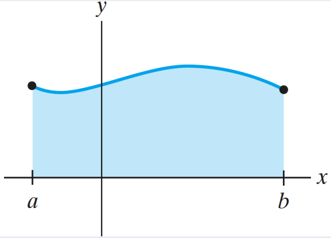
- Let f be a continuous function of one variable defined on the closed interval $[a, b]$.
- Suppose that f has only nonnegative values.
- Then, the graph of f looks like



- That f is continuous is reflected in the fact that the graph consists of an unbroken curve.
- That f is nonnegative-valued means that this curve does not dip below the x-axis.

Definite Integral of a Function of One Variable

- We know from one-variable calculus that the definite integral of f betwen a and b is denoted $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)dx}$.
- We also know that this definite integral exists and gives the area under the curve.

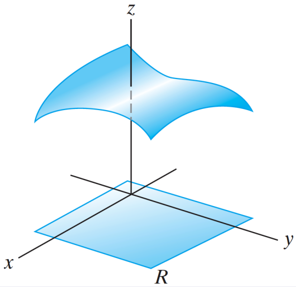


Definite Integral of a Function of Two Variables

- Now suppose that f is a continuous, nonnegative-valued function of two variables defined on the closed rectangle in $\mathbb{R}^2$.

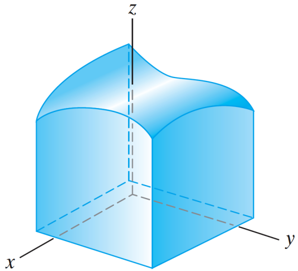
$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

- Then, the graph of f over R looks like an unbroken surface that never dips below the xy-plane



Definite Integral of a Function of Two Variables

- Analogously, there should be an integral that represents the volume under the part of the graph that lies over R



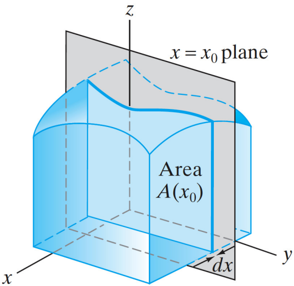
- We can find such an integral by using Cavalieri's principle.

Cavalieri's principle

Suppose f continuous nonnegative-valued on

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

- Suppose we slice by the vertical plane $x = x_0$, where x_0 is a constant between a and b

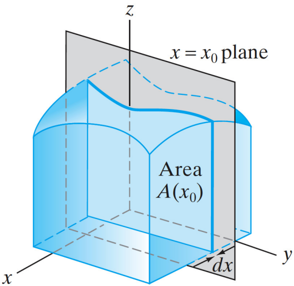


- Let $A(x_0)$ denote the cross-sectional area of such a slice

Cavalieri's principle

Suppose f is continuous nonnegative-valued on

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

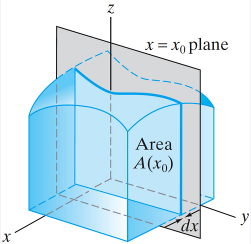


- We can think of the quantity $A(x_0)dx$ as giving the volume of an "infinitely thin" slab with:
  - Thickness dx, and
  - Cross-sectional area $A(x_0)$

Cavalieri's principle: Definite Integral of a Function of Two Variables

Suppose f continuous nonnegative-valued on

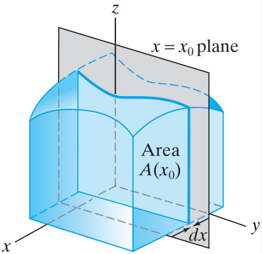
$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$



- Hence, the total volume of the solid is the "sum" of the volumes of such slabs

$$V = \int_a^b A(x)dx$$

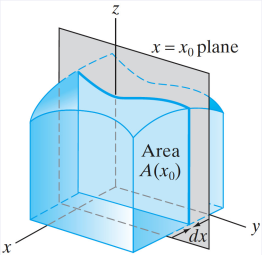
Definite Integral of a Function of Two Variables



- Note that $A(x_0)$ is the area under the curve $z = f(x_0, y)$
- This curve is obtained by slicing the surface $z = f(x, y)$ with the plane $x = x_0$
- Therefore,

$$A(x_0) = \int_c^d f(x_0, y)dy \quad (x_0 \text{ is constant})$$

Definite Integral of a Function of Two Variables



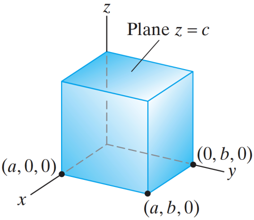
$$A(x_0) = \int_c^d f(x_0, y)dy \quad (x_0 \text{ is constant})$$

- Then,

$$V = \int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx \quad (\text{iterated integral})$$

Example 1

- Consider the case of a box



- This box is bounded
  - On top and bottom by the planes

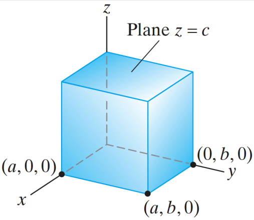
$$z = c \text{ (where } c > 0) \text{ and } z = 0$$

  - On the sides by the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, 0 \leq y \leq b\}$$

Example 1

- Consider the case of a box



- Hence, the volume of the box may be found by computing the volume under the graph of $z = c$ over the rectangle

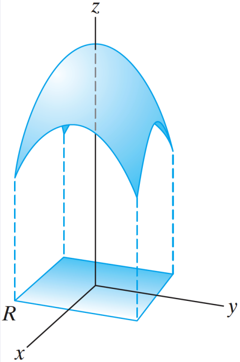
$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le a, 0 \le y \le b\}$$

- Using the iterated integral

$$V = \int_0^a \int_0^b c\,dy\,dx = \int_0^a \left(cy\Big|_{y=0}^{y=b} \right) dx = \int_0^a cb\;dx = cbx\Big|_{x=0}^{x=a} = cba$$

Example 2

- We calculate the volume under the graph of $z = 4 - x^2 - y^2$ over the square
$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$$



Example 2

- We calculate the volume under the graph of $z = 4 - x^2 - y^2$ over the square

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$$

$$\begin{aligned}
V &= \int_{-1}^{1} \int_{-1}^{1} (4 - x^2 - y^2) dy\, dx = \int_{-1}^{1} \left(4y - x^2 y - \frac{1}{3} y^3\right)\Big|_{y=-1}^{y=1} dx \\
&= \int_{-1}^{1} \left(\left(4 - x^2 - \frac{1}{3}\right) - \left(-4 + x^2 + \frac{1}{3}\right)\right) dx \\
&= \int_{-1}^{1} \left(8 - 2x^2 - \frac{2}{3}\right) dx = \left(\frac{22}{3} x - \frac{2}{3} x^3\right)\Big|_{x=-1}^{x=1} \\
&= \left(\frac{22}{3} - \frac{2}{3}\right) - \left(-\frac{22}{3} + \frac{2}{3}\right) = \frac{40}{3}
\end{aligned}$$

Proposition 1.1

- Let R be the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

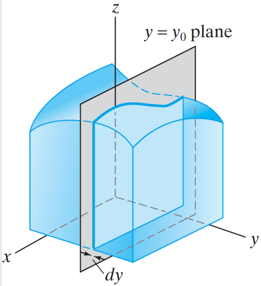
- Let f be continuous and nonnegative on R
- Then, the volume V under the graph of f over R is

$$V = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Proposition 1.1

$$V = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Remark

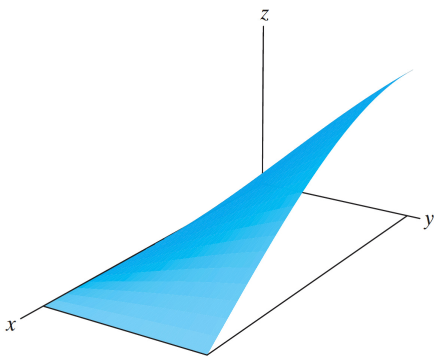


Slicing the solid with the plane $y = y_0$
instead of with the plane $x = x_0$

Example 3

- We find the volume under the graph of $z = \cos x \sin y$ over the rectangle

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{4} \right\}$$



Example 3

- We find the volume under the graph of $z = \cos x \sin y$ over the rectangle

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{4} \right\}$$

$$\begin{aligned} V &= \int_0^{\pi/2} \int_0^{\pi/4} \cos x \sin y \; dy dx = \int_0^{\pi/2} (-\cos x \cos y\ )\Big|_{y=0}^{y=\pi/4} dx \\ &= \int_0^{\pi/2} \left(\frac{-\sqrt{2}}{2} \cos x - (\cos x) \right) dx = \frac{2-\sqrt{2}}{2} \int_0^{\pi/2} \cos x \; dx \\ &= \frac{2-\sqrt{2}}{2} \sin x \Big|_0^{\pi/2} = \frac{2-\sqrt{2}}{2}(1-0) = \frac{2-\sqrt{2}}{2} \end{aligned}$$

- It is easy to check that the same result is obtained calculating

$$V = \int_0^{\pi/4} \int_0^{\pi/2} \cos x \sin y \; dx dy$$

Example 4

Does this figures haves the same volume?