Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 18

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Outline

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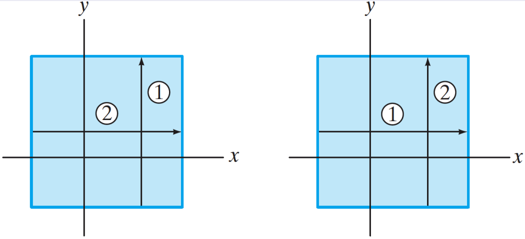
Appropriate Order of Integration for Double Integrals

- Suppose the region of integration is a rectangle
- Fubini's theorem (Theorem 2.6) says the order in which we integrate has no significance

$$\int\int_R f \, dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

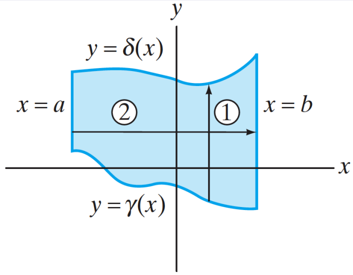


$R = [a, b] \times [c, d]$

Appropriate Order of Integration for Double Integrals

- Suppose the region is elementary of type 1 only
- We must integrate first with respect to y and then with respect to x

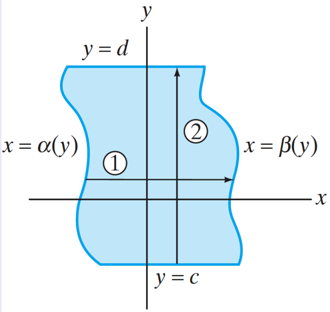
$$\int\int_D f \, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)dydx$$



Appropriate Order of Integration for Double Integrals

- Suppose the region is elementary of type 2 only
- We integrate first with respect to x and then with respect to y

$$\int\int_D f\ dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)dxdy$$



Appropriate Order of Integration for Double Integrals

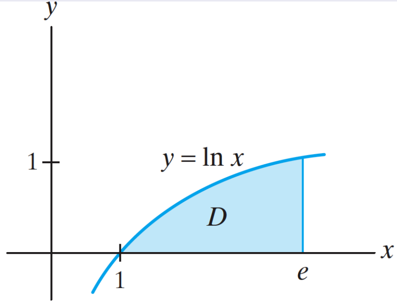
- When the region is elementary of type 3, we can **choose either order of integration**, at least in principle
- This flexibility should be used to advantage

Outline

1. Changing the Order of Integration
   - When to change
   - Examples

Example 1

- We calculate the area of the region shown in figure



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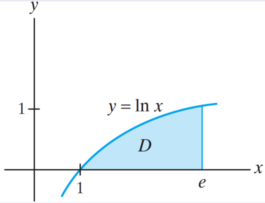

1. Considering D as a type 1 region, we obtain

$$\text{Area of } D = \int\int_D 1\, dA = \int_1^e \int_0^{\ln x} 1 dy dx = \int_1^e y\big|_0^{\ln x} dx = \int_1^e \ln x\, dx$$

This single definite integral gives the area under the graph of $y = \ln x$ over the x-interval $[1, e]$

Example 1

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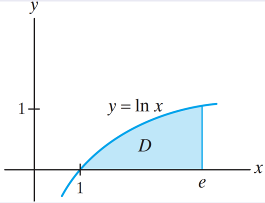

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- To evaluate this integral, we need to use integration by parts

$$\int u\, dv = uv - \int v\, du$$

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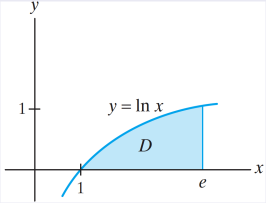

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- To evaluate this integral, we need to use integration by parts
- Let

$$u = \ln x \Rightarrow du = \frac{1}{x}\, dx \quad \text{and} \quad dv = dx \Rightarrow v = x$$

Example 1

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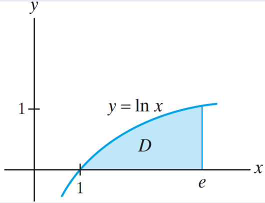

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$$= \ln x \cdot x\Big|_1^e - \int_1^e x \cdot \frac{1}{x} dx = e - 0 - \int_1^e dx = e - (e-1) = 1$$

Example 1

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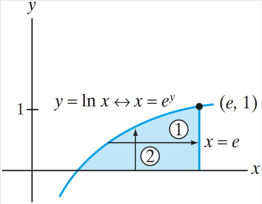


2. Integration by parts can be avoided if we integrate first with respect to x

$$\text{Area of } D = \int\int_D 1\, dA = \int_0^1 \int_{e^y}^e 1 dx dy = \int_0^1 x|_{e^y}^e dy$$
$$= \int_0^1 (e - e^y) dy = (ey - e^y)|_0^1 = (e - e) - (0 - e^0) = 1$$

Example 2

Sometimes changing the order of integration can make an impossible calculation possible

- Consider the evaluation of the following iterated integral

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)\,dx\,dy$$

- It can be shown that $\cos(x^2)$ **does not have** an antiderivative that can be expressed in terms of elementary functions.
- On the other hand, it is easy to integrate $y\cos(x^2)$ with respect to y
- This suggests finding a way to change the order of integration

Example 2

Sometimes changing the order of integration can make an impossible calculation possible

- Consider the evaluation of the following iterated integral

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)dxdy$$

- We change the order of integration in two steps:
  1. Use the limits of integration in the original iterated integral to identify the region D in $\mathbb{R}^2$

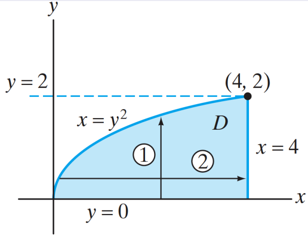
     Hopefully D turns out to be a **type 3** region

  2. Assuming that the region D in Step 1 is of type 3, change the order of integration

Example 2

- Consider the evaluation of the following iterated integral

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)dxdy$$



- The limits of integration in this particular case imply that D can be described as

$$D = \{(x, y) \mid y^2 \leq x \leq 4, 0 \leq y \leq 2\}$$

Example 2

- Consider the evaluation of the following iterated integral

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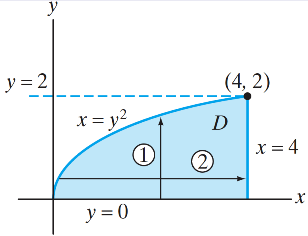

- Note that $x = y^2$ corresponds to $y = \sqrt{x}$ over this region

Example 2

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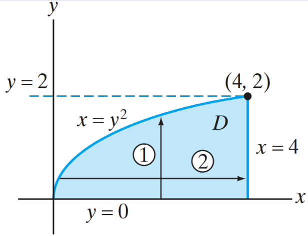

- We used this information to change the order of integration

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)dxdy = \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)dydx$$

Example 2

- Consider the evaluation of the following iterated integral

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)dxdy$$

- It is now possible to complete the calculation

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)dxdy = \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)dydx$$

$$= \int_0^4 \left(\frac{y^2}{2}\cos(x^2)\Big|_{y=0}^{y=\sqrt{x}} \right) dx = \int_0^4 \frac{x}{2}\cos(x^2)dx$$

$$u = x^2 \Rightarrow du = 2xdx$$

$$= \frac{1}{4}\int_0^{16} \cos u du = \frac{1}{4}\sin u\Big|_0^{16} = \frac{1}{4}\sin 16$$