Boundary Layer Theory
Aerodynamics

F. Mellibovsky

Escola d'Enginyeria de Telecomunicació i Aeroespacial de Castelldefels
Universitat Politècnica de Catalunya

November 13, 2018

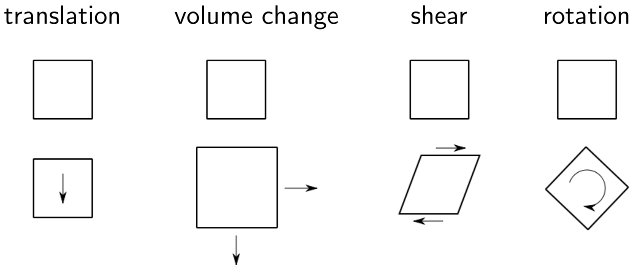
Summary

Outline

Viscous Effects in Aerodynamics

- Rate-of-change of fluid vol: $\vec{u}(\vec{r}+d\vec{r}) = \underbrace{\vec{u}(\vec{r})}_{\text{translation}} + \nabla\vec{u}\,d\vec{r} + O(\|d\vec{r}\|^2)$

$$\nabla\vec{u} = \overbrace{\underbrace{\frac{1}{3}(\nabla\cdot\vec{u})\,\vec{\vec{I}}}_{\text{volume}} + \underbrace{\left[\frac{1}{2}\left(\nabla\vec{u}+\nabla\vec{u}^T\right) - \frac{1}{3}(\nabla\cdot\vec{u})\,\vec{\vec{I}}\right]}_{\text{shear}}}^{\text{deformation}} + \underbrace{\frac{1}{2}\overbrace{\left(\nabla\vec{u}-\nabla\vec{u}^T\right)}^{(\nabla\times\vec{u})\times}}_{\text{rotation}}$$



Viscous Effects in Aerodynamics

- Rate-of-change of fluid vol: $\vec{u}(\vec{r}+d\vec{r}) = \underbrace{\vec{u}(\vec{r})}_{\text{translation}} + \nabla\vec{u}\, d\vec{r} + O(\|d\vec{r}\|^2)$

$$\nabla\vec{u} = \overbrace{\underbrace{\frac{1}{3}(\nabla\cdot\vec{u})\vec{\vec{I}}}_{\text{volume}} + \underbrace{\left[\frac{1}{2}\left(\nabla\vec{u}+\nabla\vec{u}^T\right) - \frac{1}{3}(\nabla\cdot\vec{u})\vec{\vec{I}}\right]}_{\text{shear}}}^{\text{deformation}} + \underbrace{\frac{1}{2}\overbrace{\left(\nabla\vec{u}-\nabla\vec{u}^T\right)}^{(\nabla\times\vec{u})\times}}_{\text{rotation}}$$

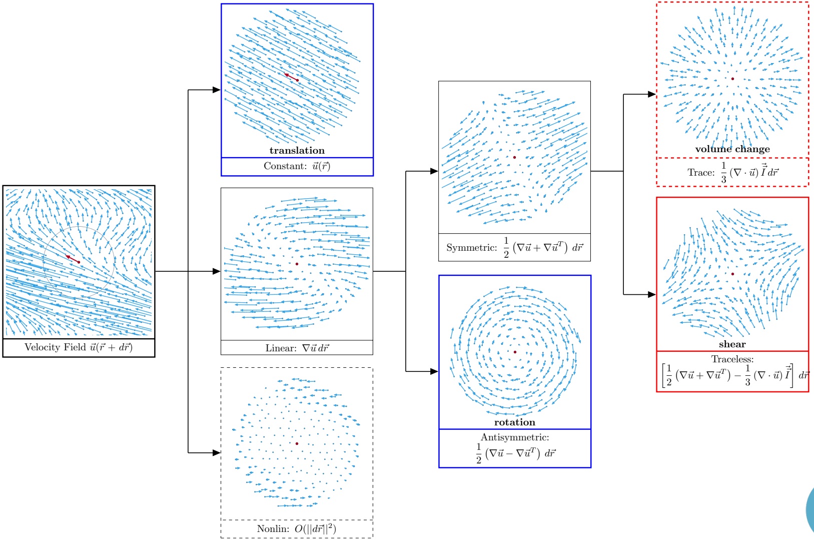
- Stress Tensor: $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_p + \vec{\vec{\tau}}$ (Isotropic newtonian fluid)
  - Elastic Stress Tensor:

$$\vec{\vec{\sigma}}_p = -\frac{1}{3}trace(\vec{\vec{\sigma}})\vec{\vec{I}} = -p\vec{\vec{I}}$$

  - Viscous Stress Tensor:

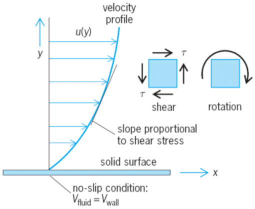
$$\vec{\vec{\tau}} = \overbrace{\cancel{\zeta(\nabla\cdot\vec{u})\vec{\vec{I}}}}^{\text{Stokes hypothesis}} + 2\mu\left(\frac{1}{2}\left(\nabla\vec{u}+\nabla\vec{u}^T\right) - \frac{1}{3}(\nabla\cdot\vec{u})\vec{\vec{I}}\right)$$

Viscous Effects in Aerodynamics

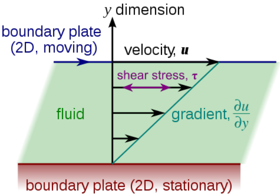


Viscous Effects in Aerodynamics

- Transverse velocity gradient:
  - shear stress:
    $$\tau = \mu \frac{\partial u}{\partial y}$$
  - Newton's law of friction
  - Generalises to Stokes' law of friction
- Viscosity:
  - physical property of the fluid
    - strongly dependent on T.
  - friction between adjacent fluid layers
  - μ: dynamic viscosity. $[\mathrm{Pa\,s}]$
  - $\nu = \mu/\rho$: kinematic viscosity. $[\mathrm{m}^2/\mathrm{s}]$

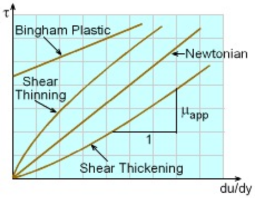


- Viscous Regimes:
  - Newtonian
  - Non-Newtonian



$$u(y) = \frac{y}{h} U$$

$$\tau_w \propto \frac{U}{h}$$



Viscous Effects in Aerodynamics

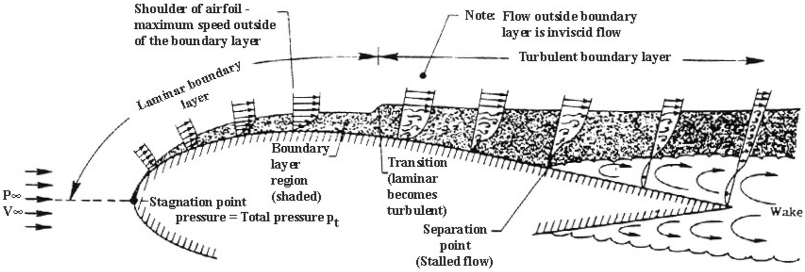
- Friction Drag
  - shear stress at walls
- Viscous Blockage
  - Pressure Drag (wake)
    - massflow reduction
  - Potential for Separation
    - momentum reduction
- Lift Production
  - due to vorticity generation[1]

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \nabla)\vec{u} + \nu \nabla^2 \vec{\omega}$$

- Potential for Turbulent Regime
  - increased friction
  - separation delayed

(1) $$\frac{d\vec{\omega}}{dt} = \frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} = (\vec{\omega} \cdot \nabla)\vec{u} - \vec{\omega}(\nabla \cdot \vec{u}) + \frac{\nabla \rho \times \nabla p}{\rho^2} + \nabla \times \left(\frac{\nabla \cdot \vec{\vec{\tau}}}{\rho} \right) + \nabla \times \vec{f}$$

Viscous Effects (Airfoil)



- Boundary layer $\longrightarrow$ LIFT and FRICTION DRAG
    - Laminar
    - Turbulent
- Wake $\longrightarrow$ FORM DRAG

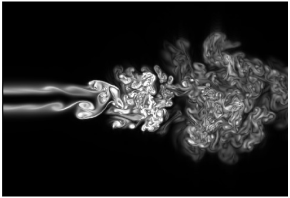
Turbulence

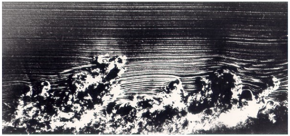
Laminar Flow:

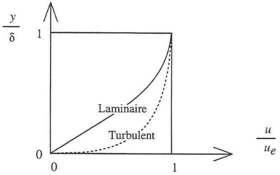
- Orderly motion of fluid particles

Turbulent Flow:

- Disorderly motion (deterministic chaos)
- Origin: Instability of the laminar flow
- Factors:
  - High Reynolds Number
  - Adverse pressure gradients
  - Wall roughness
  - Preturbulence in outer flow
  - Secondary flows
- Consequences:
  - 3D non-stationary flow
  - Efficient Mixing
  - Homogeneisation of properties
  - Increased wall fricition

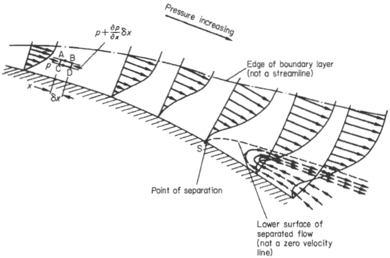


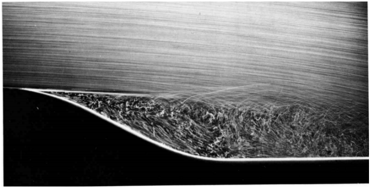


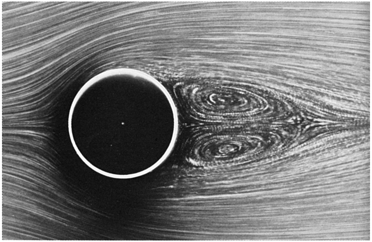


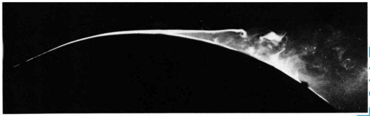
Separation

- 2D Flow:
  - Adverse pressure gradient
  - Wall friction cancels
  - Recirculated flow
- 3D Flow (e.g. swept wing):
  - Wall friction needs not cancel
  - Flow may not recirculate
  - Vortical structure formation









Turbulence-Delayed Separation

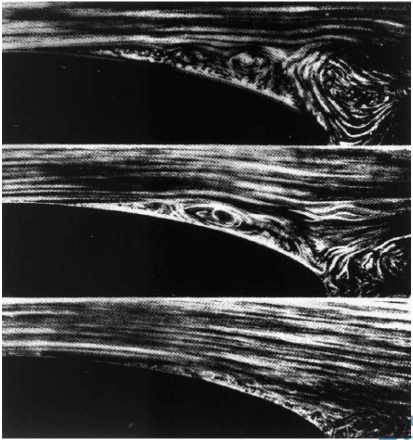
Laminar Boundary Layer:

- Low friction, but
- Little momentum
  - cannot resist strong adverse pressure gradients
  - early separation: high form drag

Turbulent Boundary Layer

- Higher friction, but
- Increased momentum due to mixing
  - better resistance to adverse pressure gradients
  - delayed separation: smaller form drag

$Re = 2, 5, 6 \times 10^4$



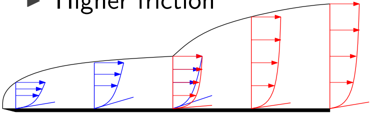
Flow Around a Streamlined Body

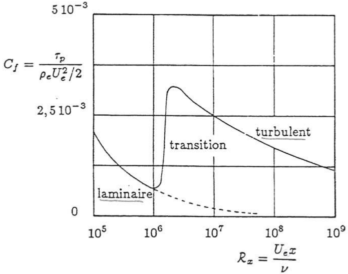
Laminar BL:

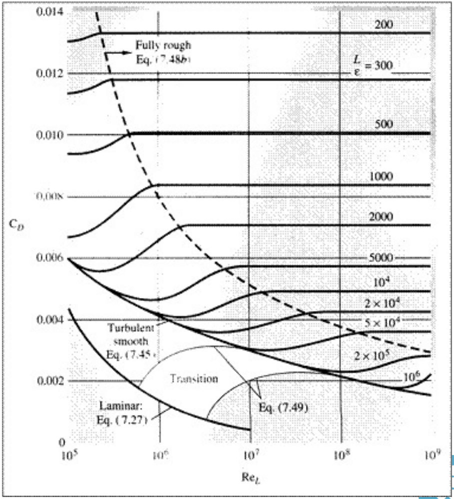
- Low friction

Turbulent BL:

- Higher friction







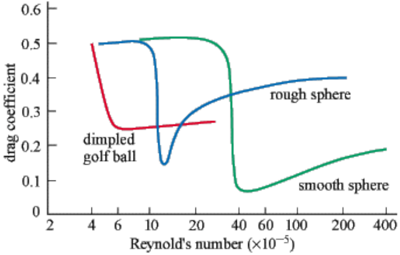
Flow Around a Sphere

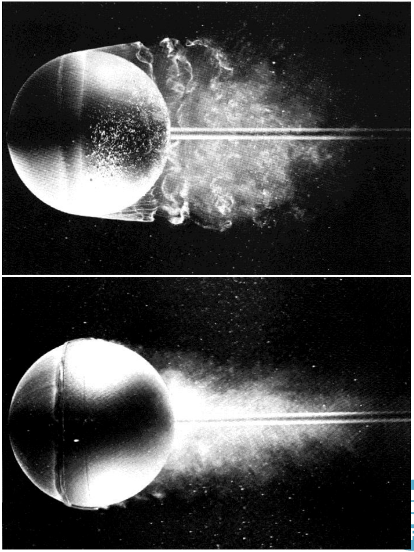
Subcritical Regime:

- Laminar BL
- Low friction / big wake

Supercritical Regime:

- Turbulent BL
- Higher friction / smaller wake

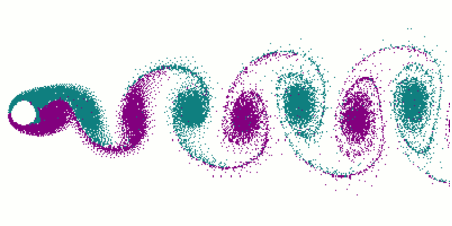


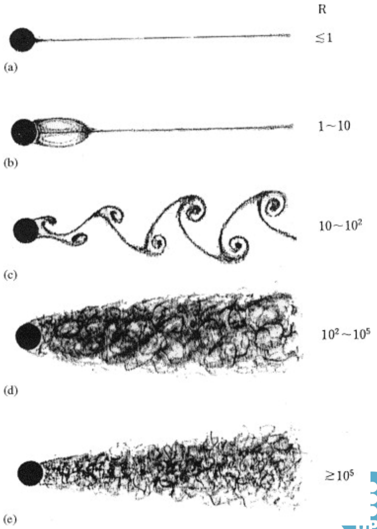


Flow Around a Cylinder

Even Greater Complexity (at low *Re*)

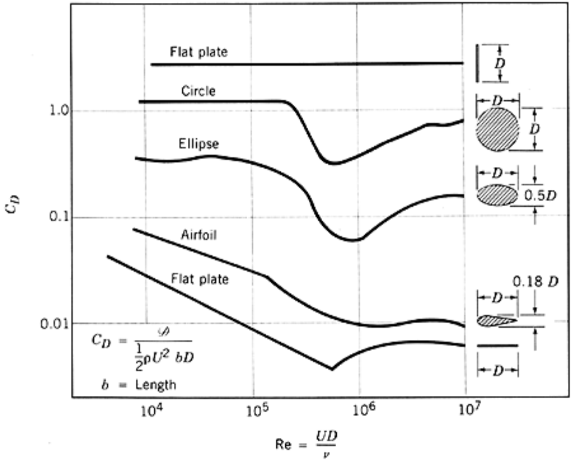
- Intermediate Laminar Regimes
- Symmetry Disruption
- Space-time Symmetries





Airfoil Drag Coefficient

What about airfoils?

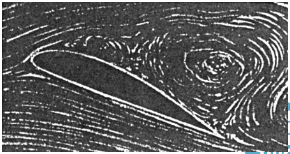


Low α:

- $\sim$ Streamlined Body
- keep laminar BL

Large α:

- $\sim$ Cylinder
- better turbulent BL



Non-Dimensional Navier-Stokes Equations

Equations in non-dimensional form:

- Mass Conservation: $\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$
- Momentum Conservation:

$$\rho\, \partial_t \vec{u} + \rho\, (\vec{u} \cdot \nabla)\, \vec{u} = \frac{-1}{\gamma M^2} \nabla p + \frac{1}{Re} \nabla \cdot \vec{\vec{\tau}} + \frac{\rho}{Fr^2} \frac{\vec{f}}{f}$$

- Energy Conservation:

$$\rho\, \partial_t e + \rho \nabla e \cdot \vec{u} =$$

$$-(\gamma - 1)\, p\, (\nabla \cdot \vec{u}) + \frac{\gamma}{Pr\, Re} \nabla \cdot (\lambda_c \nabla T) + \gamma\, (\gamma - 1)\, M^2 \left(\Theta + \frac{1}{Re} \Phi_D \right)$$

- Shear Stress Tensor: $\vec{\vec{\tau}} = \mu \left(\left(\nabla \vec{u} + \nabla \vec{u}^T \right) - \frac{2}{3} \left(\nabla \cdot \vec{u} \right) \vec{\vec{I}} \right)$; $\Phi_D = \frac{1}{2\mu} \vec{\vec{\tau}}^2$

Non-Dimensional Groupings:

- $M = \frac{u_r}{\sqrt{\gamma\, p_r / \rho_r}}$: Mach Number
- $Re = \frac{\rho_r u_r l_r}{\mu_r}$: Reynolds Number
- $Fr = \frac{u_r}{\sqrt{f\, l_r}}$: Froude Number
- $Pr = \frac{\mu_r C_p}{\lambda_{c_r}}$: Prandtl Number
- $\Theta = \frac{\phi_\tau l_r}{u_r^3}$: Non-dimensional external heat production

Properties of High-Reynolds number flows

Navier-Stokes Equations:

- Nonlinear equations, 2nd Order space derivatives
- Boundary Conditions: No-Slip at walls.

In Aerodynamics:

- High speed flow / low viscosity fluid: $Re \longrightarrow \infty$
- Consequences:
  - Euler Equations (1st Order)
    - Further Simplification (incompressibility) leads to Bernoulli Equation
  - Barotropic fluid / homentropic flow + conservative body force:
    - Irrotational Flow $\longrightarrow$ Scalar Potential for velocity
  - Potential flows:
    - Incompressible flow
    - Compressible Subcritical flow
    - Supercritical and Supersonic flows (attached shock waves), but only approximately

Shortcomings of Potential Flow Theory

Shortcomings:

- No-slip boundary conditions not applicable!
- Irrotational flow: no vorticity $\longrightarrow$ no circulation $\longrightarrow$ no Lift!
- Aerodynamic lift artificially introduced through an arbitrary circulation (Kutta condition)
  - Vortex potential: irrotational except at singularity $\longrightarrow$ circulation.
- No aerodynamic drag force predicted in many cases (d'Alembert's paradox for 2D flows) or seriously underestimated.
  - Lift-induced drag in 3D.
- Several observed phenomena cannot be explained or predicted:
  - turbulence
  - wakes
  - detached/separated flow

Outline

Boundary Layer Hypothesis

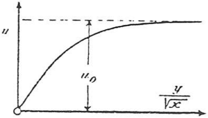
L. Prandtl (1904): *Über Flüssigkeitsbewegungen bei sehr kleiner Reibung (Fluid Flow in Very Little Friction)*

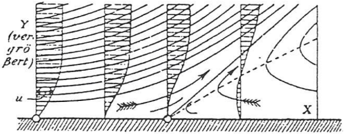
In most of the fluid domain:

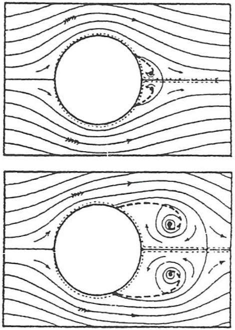
- Inviscid flow remains a valid hypothesis

Close to walls:

- Recover the no-slip boundary condition by considering the effects of viscosity.







Mechanisms for Momentum Transport

Advection:

- Transport along streamlines
- Characteristic Time: $t_a \sim \frac{L}{U}$

Diffusion

- Transport across streamlines
- Interaction at the molecular level through viscosity
- Characteristic Time: $t_v \sim \frac{\delta^2}{\nu}$

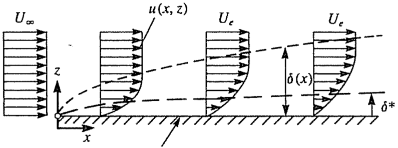
Comparison: $\frac{t_v}{t_a} \sim \left(\frac{\delta}{L}\right)^2 Re_L$

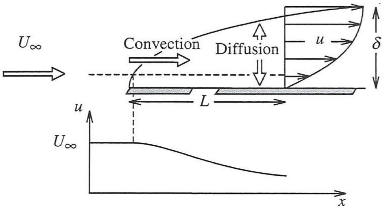
Viscosity affects a thin layer such that $t_v \sim t_a$ (BL Hypothesis):

$$\delta^* = \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} = -\cancel{\frac{1}{\rho}\frac{\partial p}{\partial x}} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$





Dimensional Analysis

Nondimensionalising with L and U (ρ and μ constants)$^{(*)}$:

- Continuity:

$$\underbrace{\frac{\partial u^*}{\partial x^*}}_{O(1)} + \frac{\partial v^*}{\partial y^*} = 0 \longrightarrow v^* \sim \delta^*$$

- x-momentum:

$$\underbrace{\frac{\partial u^*}{\partial t^*}}_{O(1)} + \underbrace{u^* \frac{\partial u^*}{\partial x^*}}_{O(1)} + \underbrace{v^* \frac{\partial u^*}{\partial y^*}}_{O(\delta^* \cdot \frac{1}{\delta^*})} = -\underbrace{\frac{\partial p^*}{\partial x^*}}_{O(1)} + \underbrace{\frac{1}{Re_L}}_{O(\delta^{*2})} \left[\underbrace{\frac{\partial^2 u^*}{\partial x^{*2}}}_{O(1)} + \underbrace{\frac{\partial^2 u^*}{\partial y^{*2}}}_{O(\frac{1}{\delta^{*2}})} \right]$$

- y-momentum:

$$\underbrace{\frac{\partial v^*}{\partial t^*}}_{O(\delta^*)} + \underbrace{u^* \frac{\partial v^*}{\partial x^*}}_{O(\delta^*)} + \underbrace{v^* \frac{\partial v^*}{\partial y^*}}_{O(\delta^* \cdot \frac{\delta^*}{\delta^*})} = -\underbrace{\frac{\partial p^*}{\partial y^*}}_{O(\frac{1}{\delta^*})} + \underbrace{\frac{1}{Re_L}}_{O(\delta^{*2})} \left[\underbrace{\frac{\partial^2 v^*}{\partial x^{*2}}}_{O(\delta^*)} + \underbrace{\frac{\partial^2 v^*}{\partial y^{*2}}}_{O(\frac{1}{\delta^*})} \right]$$

$^{(*)}$Similar analyses hold for compressible flow and for variable viscosity

2D Boundary Layer Equations

For Stationary, incompressible flow:

- Continuity:

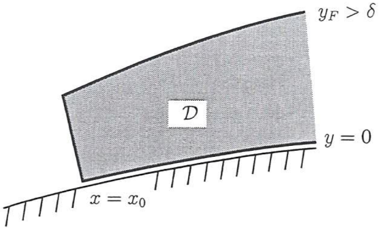
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- y-momentum$^{(*)}$:

$$\frac{\partial p}{\partial y} = 0$$

- x-momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$



With boundary conditions:

- Wall: $u(x,0) = v(x,0) = 0$
- B.L. edge: $u(x,\infty) \longrightarrow u_e(x)$

$$\rho_e u_e \frac{du_e}{dx} = -\frac{dp}{dx}$$

- On $x = x_0$: $u(x_0, y) = u_0(y)$

$^{(*)}$ Assuming small curvature: $\delta << R(x) = 1/\kappa(x)$

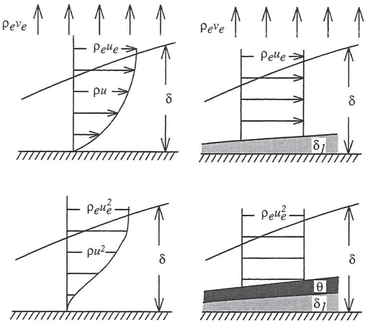
NS: elliptic equations; BL: parabolic equations: v_e cannot be imposed!

Boundary Layer Characteristic Properties

- Wall Shear Stress:

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_w$$

real flow equivalent flow



- Skin Friction Coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$$

- Displacement Thickness:

$$\delta_1 = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e} \right) dy$$

- Momentum Thickness:

$$\delta_2 = \theta = \int_0^\delta \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e} \right) dy$$

- Form Factor:

$$H = \frac{\delta_1}{\theta}$$

Boundary Layer Integral Equations

Integration with respect to y yields 1-dimensional equations:

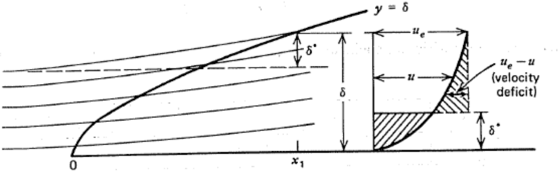
$$\int_0^{\delta} [Eq]\, dy$$

- Continuity:

$$\frac{1}{u_e}\frac{d}{dx}\left(u_e\left(\delta-\delta_1\right)\right)=\frac{d\delta}{dx}-\frac{v_\delta}{u_e}=C_E$$

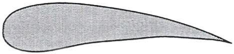
- Momentum:

$$\frac{d\theta}{dx}+\theta\left(\frac{H+2}{u_e}\frac{du_e}{dx}\right)=\frac{C_f}{2}$$



Solution Methods

- Solve the Navier-Stokes Equations with no assumptions
- Exploit that $Re_L >> 1$
  - Inviscid Flow – Boundary Layer Coupling
    1. Solve Euler Equations around airfoil $\longrightarrow p(x)$
    2. Solve Boundary Layer Equations $\longrightarrow u, v(x, y) \longrightarrow \delta_1(x), \tau_w(x)$
    3. Solve Euler Equations around modified airfoil and wake $\longrightarrow$ new $p(x)$



  - Airfoil Performance Calculations:
    - Lift: integration of $p(x)$ from step 3
    - Form Drag: integration of $p(x)$ from step 3
    - Friction Drag: integration of $\tau_w(x)$ from step 2
  - Options for step 2:
    - Numerical solution of the local equations
    - Self-Similar Solutions (Falkner-Skan)
    - Integral Methods (approximate)

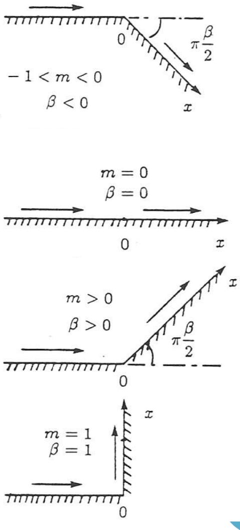
Falkner-Skan Self-Similar Solutions

Wedge Flows:

- Complex Potential: $F(z) = \frac{k}{m+1} z^{m+1}$
  - $\psi(r, \theta) = \mathrm{Im}[F] = \frac{k}{m+1} r^{m+1} \sin((m+1)\theta)$
- Complex Velocity: $w^*(z) = \frac{dF}{dz} = kz^m$
  - $u_r(r, \theta) = \frac{1}{r}\frac{d\psi}{d\theta} = kr^m \cos((m+1)\theta)$
- Streamlines through origin: $\theta_w = \frac{n\pi}{m+1}$
  - interpret as walls: $u_r(r, \theta_w) = kr^m (-1)^n$
- Wall velocity: $u_e(x) = kx^m$
- Deflection Angle:

$$\alpha = \frac{\pi\beta}{2} \qquad \beta = \frac{2m}{m+1}$$

- Cases:
  - decelerated flow: $-0.5 \leq m < 0$
  - flate plate: $m = 0$
  - accelerated flow: $0 \leq m < 1$
  - stagnation point: $m = 1$



Falkner-Skan Self-Similar Solutions

Non-dimensional self-similar variables:

$$\frac{u}{u_e} = f'(\eta) \qquad \xi = x \;\text{ and }\; \eta = \frac{y}{g(x)}$$

With the scaling:

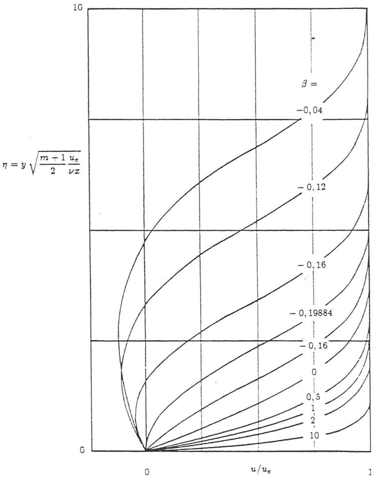
$$g(x) = \sqrt{\frac{2}{m+1}\,\frac{\nu\, x}{u_e}} \sim \delta(x)$$

The Boundary Layer equations read:

$$f''' + f\, f'' + \beta\left(1 - f'^2\right) = 0$$

with boundary conditions:

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1$$



Falkner-Skan Self-Similar Solutions

- Wall Friction Factor:

$$\frac{\tau_w}{\rho_e u_e^2}\sqrt{\frac{u_e x}{\nu}} = \frac{C_f}{2}\sqrt{Re_x} = f''(0)\sqrt{\frac{m+1}{2}}$$

- Displacement Thickness:

$$\frac{\delta_1}{x}\sqrt{Re_x} = I_1\sqrt{\frac{2}{m+1}} \qquad \text{with} \qquad I_1 = \int_0^\infty (1-f')d\eta$$

- Momentum Thickness:

$$\frac{\theta}{x}\sqrt{Re_x} = I_2\sqrt{\frac{2}{m+1}} \qquad \text{with} \qquad I_2 = \int_0^\infty f'(1-f')d\eta$$

- Form Factor:

$$H = \frac{I_1}{I_2}$$

Falkner-Skan Self-Similar Solutions

	β	m	$\frac{Re_\theta}{\sqrt{Re_x}}$	$\frac{Re_{\delta_1}}{\sqrt{Re_x}}$	H	$\frac{C_f}{2}\sqrt{Re_x}$	$\frac{C_f}{2}Re_\theta$	$\frac{\theta Re_\theta}{u_e}\frac{du_e}{dx}$	$Re_\theta \frac{d\theta}{dx}$
	20.00000	-1.11111	0.35112	0.73026	2.07979	1.22111	0.42876	0.13698	-0.13014
	10.00000	-1.25000	0.32592	0.68101	2.08950	1.29939	0.42350	0.13278	-0.11950
	2.00000	∞	0.00000	0.00000	2.15541	∞	0.38938	0.10652	-0.05326
SP	1.00000	1.00000	0.29234	0.64789	2.21623	1.23259	0.36034	0.08546	0.00000
	0.50000	0.33333	0.42899	0.98537	2.29694	0.75745	0.32494	0.06134	0.06134
	0.28571	0.16666	0.50895	1.20511	2.36781	0.58255	0.29649	0.04317	0.10793
FP	0.00000	0.00000	0.66411	1.72079	2.59110	0.33206	0.22052	0.00000	0.22052
	-0.04000	-0.01961	0.69419	1.84404	2.65639	0.29052	0.20168	-0.00945	0.24567
	-0.08000	-0.03846	0.72786	1.99731	2.74409	0.24512	0.17841	-0.02038	0.27508
	-0.12000	-0.05660	0.76628	2.20057	2.87177	0.19351	0.14828	-0.03324	0.31021
	-0.16000	-0.07407	0.81115	2.50823	3.09067	0.12981	0.10535	-0.04879	0.35370
Sep	-0.19884	-0.09043	0.86811	3.49779	4.02923	0.00000	0.00000	-0.06815	0.41088
	-0.16000	-0.07407	0.76792	5.18496	6.75200	-0.08544	-0.06561	-0.04368	0.31669
	-0.12000	-0.05660	0.63692	6.40508	10.05630	-0.09817	-0.06253	-0.02296	0.21432
	-0.08000	-0.03846	0.47987	7.90226	16.46750	-0.09169	-0.04400	-0.00886	0.11957
	-0.04000	-0.01961	0.28891	10.38459	35.94357	-0.06766	-0.01955	-0.00164	0.04255

- SP: Stagnation Point
- FP: Flat Plate
- Sep: Separation

$$Re_x = \frac{u_e x}{\nu} \qquad Re_{\delta_1} = \frac{u_e \delta_1}{\nu} \qquad Re_\theta = \frac{u_e \theta}{\nu}$$

Two Useful Cases

Flat Plate Boundary Layer (Blasius)

- Wall Friction Factor:

$$\tau_w = 0.332\, \rho_e u_e^2\, Re_x^{-1/2}$$

- Displacement Thickness:

$$\delta_1 / x = 1.7208\, Re_x^{-1/2}$$

- Momentum Thickness:

$$\theta / x = 0.664\, Re_x^{-1/2}$$

- Form Factor: $H = 2.591$
- Friction force: $F/b = \rho_e u_e^2 \theta(L)$

2D Stagnation Point ($u_e = kx$)

- Wall Friction Factor:

$$\tau_w = 1.2326\, \rho_e\, kx \sqrt{k\nu}$$

- Displacement Thickness:

$$\delta_1 = 0.6479 \sqrt{\nu / k}$$

- Momentum Thickness:

$$\theta = 0.2923 \sqrt{\nu / k}$$

- Form Factor: $H = 2.2162$
- $\tau_w = 0$ at stagnation point

Integral Method (Only approximate)

Open Problem:

- 1 equation:

$$\frac{d\theta}{dx} + \theta \left(\frac{H+2}{u_e} \frac{du_e}{dx} \right) = \frac{C_f}{2}$$

- 3 unknowns: θ, H and C_f

Closure: Assume Flat Plate laws hold (approximate)

- Assumptions:

$$\frac{C_f}{2} Re_\theta = b = 0.2205 \qquad \text{and} \qquad H = 2.591$$

- Approximate equation:

$$\frac{d\theta}{dx} + \theta \left(\frac{H+2}{u_e} \frac{du_e}{dx} \right) = \frac{b\nu}{u_e \theta}$$

- Solution:

$$\left[\theta u_e^{(H+2)} \right]_{x_1}^2 = \left[\theta u_e^{(H+2)} \right]_{x_0}^2 + 2b \int_{x_0}^{x_1} \frac{u_e^{2(H+2)}}{u_e/\nu} dx$$

Integral Method (Only approximate)

Alternative closure:

- Pohlhausen's polynomial approximation (4th order):

$$\frac{u}{u_e} \equiv f(\eta) \ \text{with} \ \eta = \frac{y}{\delta}$$

  - BC at $\eta = 0$: $f(0) = 0$
  - BC at $\eta = 1$: $f(1) = 1$ and $f'(1) = 0$
  - Analyticity across $\eta = 1$: $f''(1) = f'''(1) = f^{iv}(1) = 0$
  - from BL equation at $\eta = 0$: $f''(0) = -\frac{\delta^2}{\nu}\frac{du_e}{dx} \equiv -\Lambda$
  - from BL' equation at $\eta = 0$: $f'''(0) = 0$

$$f(\eta) = 2\eta - 2\eta^3 + \eta^4 + \frac{1}{6}\Lambda\eta(1-\eta)^3$$

  - Cases:
    - $\Lambda = 0$: Flat plate boundary layer.
    - $\Lambda = 12$: First overshooting velocity profile.
    - $\Lambda = -12$: Separation profile.

Integral Method (Only approximate)

- Using this profile:
  - $\frac{C_f}{2} = \frac{\nu}{u_e \delta}\left(2 + \frac{1}{6}\Lambda\right)$
  - $\delta_1 = \delta\left(\frac{3}{10} - \frac{1}{120}\Lambda\right)$
  - $\theta = \frac{\delta}{315}\left(37 - \frac{1}{3}\Lambda - \frac{5}{144}\Lambda^2\right)$
- Substitute in momentum integral equation and solve for Λ:

$$\frac{d\Lambda}{dx} = \frac{1}{u_e}\frac{du_e}{dx}\frac{-90720 - 10512\Lambda + 282\Lambda^2 - 10\Lambda^3}{-5328 + 48\Lambda + 5\Lambda^2} + \frac{\frac{d^2u_e}{dx^2}}{\frac{du_e}{dx}}\frac{\Lambda}{2}$$

- Method:
  - Pick $\frac{du_e}{dx}$ from inviscid calculation.
  - Choose initial value for δ (or θ).
  - Integrate to find evolution of Λ.
  - Obtain the boundary layer thickness δ.

Integral Method (Only approximate)

Alternative closure:

- Thwaites' correlation method
  - exploit correlation of analytical and experimental results.
  - $H = H(\lambda)$ and $\frac{C_f}{2} Re_\theta \equiv l = l(\lambda)$
    - with $\lambda = \frac{\theta^2}{\nu}\frac{du_e}{dx}$, a nondimensional pressure gradient parameter
  - The momentum integral equation becomes:

$$\frac{u_e}{\nu}\frac{d\theta^2}{dx} = 2\left[l(\lambda) - \lambda\left(H(\lambda)+2\right)\right] \equiv F(\lambda) \approx 0.45 - 6.0\lambda$$

  - resulting, after integration, in [Cebeci & Bradshaw]

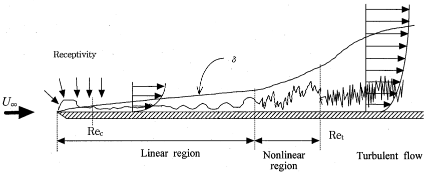
$$\begin{aligned}
\frac{\theta^2 u_e^6}{\nu} &= 0.45\int_0^x u_e^5 dx + \left(\frac{\theta^2 u_e^6}{\nu}\right)_0 \\
l &= \begin{cases} 0.22 + 1.57\lambda - 1.80\lambda^2 & \text{for} \quad 0 < \lambda < 0.1 \\ 0.22 + 1.402\lambda + 0.018\frac{\lambda}{\lambda+0.107} & \text{for} \quad -0.1 < \lambda < 0 \end{cases} \\
H &= \begin{cases} 2.61 - 3.75\lambda + 5.24\lambda^2 & \text{for} \quad 0 < \lambda < 0.1 \\ 2.088 + \frac{0.0731}{\lambda+0.14} & \text{for} \quad -0.1 < \lambda < 0 \end{cases}
\end{aligned}$$

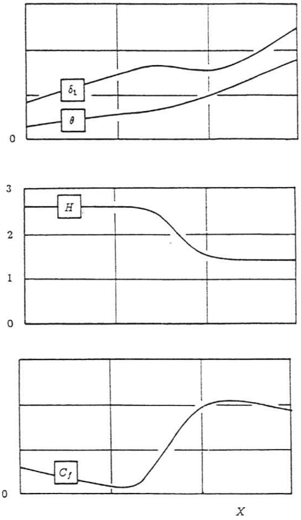
Outline

Boundary Layer Laminar-Turbulent Transition

Typical Transition Scenario:

- Laminar flow over a short distance
- Instability to small perturbations
  - Tollmien-Schlichting waves
- Linear growth of waves
- Nonlinear saturation
- Turbulence





Factors & Properties of Turbulent Flow

Factors influencing transition:

- Reynolds number
- External perturbations:
  - Preturbulence levels
  - wall roughness
- Pressure gradient (2D)
- Transverse flow (3D)
- Wall suction
- Others: compressibility, wall curvature, heat transfer...

Transition prediction:

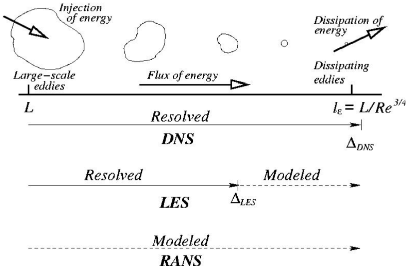
- No general method
  - Different Criteria

Properties of Turbulent Flow:

- High Reynolds numbers
- Mixing capabilities
  - better than viscous diffusion
  - homogeneisation
- faster BL thickness growth
- Flow is intrinsically 3D
- Flow is intrinsically time-dependent
- High energy dissipation
- Navier-Stokes hold (turbulent scales $>>$ mollecular scales)
- Deterministic chaos

Methods for Solving Turbulent Flows

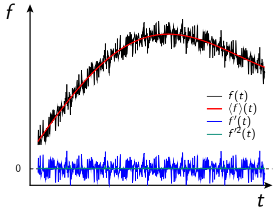
- Direct Navier Stokes (DNS)
- Reynolds Averaging (RANS)
  - Solve transport equation for mean flow
  - All turbulent scales are modeled
- Filtering (LES)
  - Solve transport equation for resolvable scales
  - Large turbulent scales (large eddies) are solved
  - Small turbulent scales are modeled



Reynolds Averaged Navier Stokes (RANS)

Decompose field f in the sum of an average $\overline{f}$ (filtering or ensemble averaging) plus a fluctuation f'

$$f = \overline{f} + f'$$



with

$$\overline{f} = \frac{1}{\Delta t}\int_t^{t+\Delta t} f \; dt \qquad \text{or} \qquad \overline{f} = \lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N} f_i \qquad \text{and} \qquad \overline{f'} = 0$$

Some averaging properties (f, g turbulent and α non-turbulent fields):

$$\begin{aligned} \overline{f+g} &= \overline{f} + \overline{g} \\ \overline{f\,g} &= \overline{f}\,\overline{g} + \overline{f'g'} \\ \overline{\alpha\, f} &= \alpha\overline{f} \end{aligned} \qquad\qquad \begin{aligned} \overline{\frac{\partial f}{\partial t}} &= \frac{\partial \overline{f}}{\partial t} \\ \overline{\frac{\partial f}{\partial x_i}} &= \frac{\partial \overline{f}}{\partial x_i} \end{aligned}$$

For compressible flows: (density-weighted) Favre averaging

$$\widetilde{f} = \overline{\rho f}/\overline{\rho} \qquad \text{and} \qquad f' = f - \widetilde{f}$$

Reynolds Averaged Navier Stokes (RANS)

- Mass Conservation:

$$\frac{\partial \overline{u_j}}{\partial x_j} = 0$$

- Momentum Conservation:

$$\rho \frac{\partial \overline{u_i}}{\partial t} + \rho \overline{u_j} \frac{\partial \overline{u_i}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \overline{u_i}}{\partial x_j} - \rho \, \overline{u_i' u_j'} \right)$$

- Turbulent Fluctuations Kinetic Energy Equation:

$$\rho \frac{\partial k}{\partial t} + \overbrace{\rho \, \overline{u_j} \frac{\partial k}{\partial x_j}}^{\text{transport}} = \overbrace{-\rho \, \overline{u_i' u_j'} \, s_{ij}}^{\text{production}} - \overbrace{2 \, \mu \, \overline{s_{ij}' s_{ij}'}}^{\text{dissipation}} + \overbrace{\frac{\partial}{\partial x_j} \left(-\overline{p' u_j'} + 2 \, \mu \, \overline{u_i' s_{ij}'} - \frac{\rho}{2} \, \overline{u_i'^2 u_j'} \right)}^{\text{diffusion}}$$

$$= -\rho \, \overline{u_i' u_j'} \, s_{ij} - \mu \, \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_j} \left(-\overline{p' u_j'} + \mu \frac{\partial k}{\partial x_j} - \frac{\rho}{2} \, \overline{u_i'^2 u_j'} \right)$$

with $k = \frac{\overline{u_i'^2}}{2}$ and $s_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

Reynolds Averaged Navier Stokes (RANS)

Open Problem:

- 4 equations:
  - mass
  - momentum (3)
- 10 unknowns:
  - $\overline{p}$
  - $\overline{u_i}$ ($\overline{u}$, $\overline{v}$, $\overline{w}$)
  - 6 Reynolds stresses: $\rho\,\overline{u_i' u_j'}$

$$\vec{\vec{\tau}}_t = -\rho\overline{u_i' u_j'} = -\rho \begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{pmatrix}$$

Closure:

- Reynolds Stress Model
- Turbulent Viscosity Models:

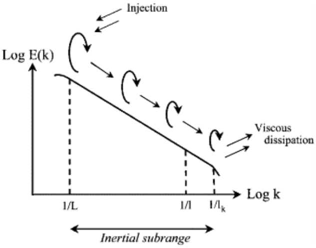
$$-\overline{u_i' u_j'} \equiv \nu_t \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij}$$

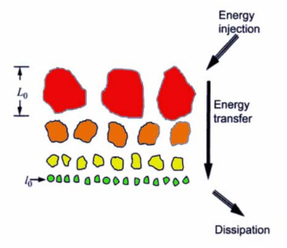
  - Algebraic:
    - Mixing length (Baldwin-Lomax, Cebeci-Smith...)
  - 1 transport equation models:
    - Prandtl's one-equation
    - Spallart-Almaras
    - Baldwin-Barth
  - 2 transport equations models:
    - k-ϵ: Std, RNG, Realizable: $\nu_t = C_\mu \dfrac{k^2}{\epsilon}$
    - k-ω: Std, SST ($\omega \sim \epsilon / k$): $\nu_t = \gamma \dfrac{k}{\omega}$
- Nonlinear Eddy Viscosity Models:
  - Explicit nonlinear contitutive relation
  - v2-f models ($\overline{v^2} - f$, $\zeta - f$)

Alternatives to RANS

Scale Resolving Simulations (SRS): Small scale filtering

- Approach:
  - Solve large turbulent scales
  - Model small turbulent scales
- Methods:
  - Large Eddy Simulation (LES)
  - Hybrid RANS-LES Models
    - Dettached Eddy Simulation (DES)
    - Scale Adaptive Simulation (SAS)





2D Turbulent Boundary Layer Equations

For 2D, Stationary, incompressible flow:

- Continuity:

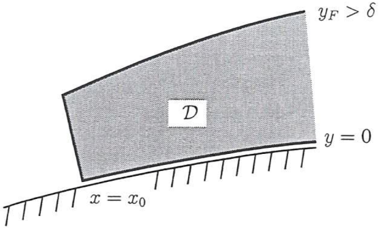
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- y-momentum$^{(*)}$:

$$\frac{\partial p}{\partial y} = 0$$

- x-momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} - \rho\, \overline{u'v'}\right)$$



With boundary conditions:

- Wall: $u = v = \overline{u'v'} = 0$
- B.L. edge: $u \rightarrow u_e$ and $\overline{u'v'} \rightarrow 0$

$$\rho_e u_e \frac{du_e}{dx} = -\frac{dp}{dx}$$

- On $x = x_0$: laminar BL results

Assuming $\overline{u'w'} = \overline{v'w'} = 0$

$^{(*)}$ Assuming small curvature: $\delta << R(x) = 1/\kappa(x)$

NS: elliptical equations; BL: parabolic equations: v_e cannot be imposed!

Turbulent BL Solution Methods

- Solve Local Equations
  - Needs turbulent model for closure
- Self-Similar Solutions
  - Universal profile close to the wall (viscous sublayer)
  - Self-similar log-law in most of the boundary layer
- Integral Methods
  - Integral equations formally identical to laminar case
  - Experimental input is needed

Experimental Results for the Flat Plate Boundary Layer:

- $H \to 1$ as $Re \to \infty$ ($H \sim 1.3, 1.4$ for $Re \sim 10^6, 10^7$)
- Empirical law for fricition factor:

$$C_f = \frac{0.0368}{Re_x^{1/6}} \quad \xrightarrow{\frac{d\theta}{dx}=\frac{C_f}{2}} \quad \frac{\theta}{x} = \frac{0.0221}{Re_x^{1/6}} \qquad C_f = \frac{0.0172}{Re_\theta^{1/5}}$$

Integral Method (Laminar and Turbulent BL)

Open Problem:

- 1 equation:

$$\frac{d\theta}{dx} + \theta\left(\frac{H+2}{u_e}\frac{du_e}{dx}\right) = \frac{C_f}{2}$$

- 3 unknowns: θ, H and C_f

Closure: Assume Flat Plate laws hold (approximate)

- Assumptions:

$$\frac{C_f}{2} = \frac{b}{Re_\theta^{m_0}} \qquad \text{and} \qquad H \text{ constant}$$

- with

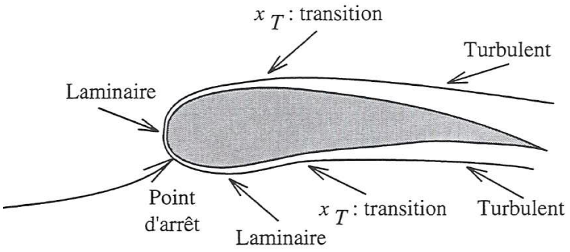
	m_0	b	H
laminar	1	0.2205	2.591
turbulent	1/5	0.0086	1.4

- Solution:

$$\left[\theta u_e^{(H+2)}\right]_{x_1}^{m_0+1} = \left[\theta u_e^{(H+2)}\right]_{x_0}^{m_0+1} + (m_0+1)b\int_{x_0}^{x_1}\frac{u_e^{(m_0+1)(H+2)}}{(u_e/\nu)^{m_0}}dx$$

Integral Method (Flow Around Airfoil)

- Set $x_0 = 0$ at the stagnation point
  - Assume laminar BL at x_0
  - Use Falkner-Skan solution ($m = 1$) to compute $\theta(0)$
- Use laminar integral method from x_0 to x_T
- x_T, the transiton point, is:
  - known or assumed
  - determined with some transition criterion
- Assume continuity of θ at x_T: $\theta_{lam}(x_T) = \theta_{turb}(x_T)$
- Use turbulent integral method beyond x_T



Integral Method (Results For Flat Plate)

Assuming the BL starts directly at the leading edge:

$$p = \frac{k_1}{Re_x^{s_0}} \qquad p = \frac{k_2}{Re_\theta^{m_0}}$$

	Laminar $m_0 = 1$ $s_0 = 1/2$		Turbulent $m_0 = 1/5$ $s_0 = 1/6$	
p	k_1	k_2	k_1	k_2
θ/x	0.664	0.441	0.0221	0.0103
δ_1/x	1.721	1.143	0.0309	0.0144
C_f	0.664	0.441	0.0368	0.0172

Relations for turbulent flows:

$$\left.\begin{aligned} H &= 1.4 \\ C_f &= \frac{0.0368}{Re_x^{1/6}} \end{aligned}\right\} \xrightarrow{\frac{d\theta}{dx}=\frac{C_f}{2}} \frac{\theta}{x}(Re_x) \longrightarrow \left\{\begin{aligned} &\frac{\theta}{x}(Re_\theta) \\ &C_f(Re_\theta) \end{aligned}\right\} \xrightarrow{\delta_1 = \theta H} \left\{\begin{aligned} &\frac{\delta_1}{x}(Re_x) \\ &\frac{\delta_1}{x}(Re_\theta) \end{aligned}\right.$$

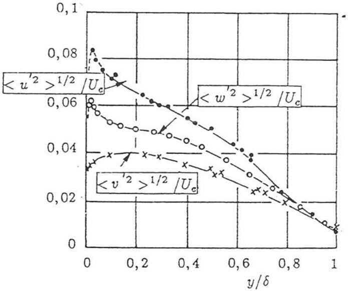
Turbulent BL Structure

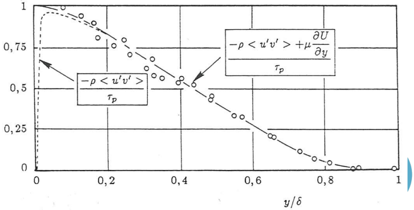
- fluctuation velocities self-correlation $\overline{u_i'^2}$
  - $\overline{u_i'^2} = 0$ at the wall
  - Rapid increase to a maximum
  - Slow decrease to outer flow levels
  - $\overline{u'^2} > \overline{w'^2} > \overline{v'^2}$
- friction stress $\tau = \mu \dfrac{\partial u}{\partial y} - \rho\, \overline{u'v'}$
  - Viscous Sublayer: $\mu \dfrac{\partial u}{\partial y}$ dominates
  - Log layer: $-\rho\, \overline{u'v'}$ dominates
- velocity and length scales:
  - Shear (or friction) velocity and length:

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} \qquad l_\tau = \frac{\nu}{u_\tau}$$

  - Wall variables:

$$u^+ = \frac{u}{u_\tau} \qquad y^+ = \frac{y\, u_\tau}{\nu}$$





Mixing Length Model

Prandtl (1925): A fluid particle with mean longitudinal velocity $u(y)$, and transversally displaced by the transversal veocity fluctuation v' retains its momentum over a length l. The longitudinal velocity fluctuation is then $u' = u(y + l) - u(y) \simeq l \frac{\partial u}{\partial y}$. We then assume $v' \simeq -u'$

- Model for Reynold-Stresses:

$$-\overline{u'v'} = F^2 l^2 \left(\frac{\partial u}{\partial y}\right)^2$$

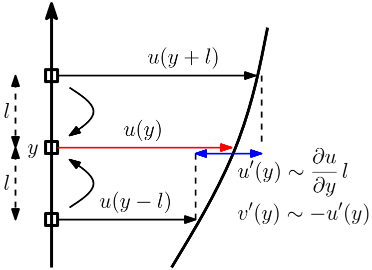
- Experimental Mixing Length:

$$\frac{l}{\delta} = 0.085 \tanh\left(\frac{\chi}{0.085}\frac{y}{\delta}\right)$$

  - $\chi = 0.41$ von Karman constant

- Damping Function:

$$F = 1 - exp\left(-\frac{y^+}{A^+}\right); \; A^+ = 26$$



Mixing Length Model

Prandtl (1925): A fluid particle with mean longitudinal velocity $u(y)$, and transversally displaced by the transversal veocity fluctuation v' retains its momentum over a length l. The longitudinal velocity fluctuation is then $u' = u(y + l) - u(y) \simeq l \frac{\partial u}{\partial y}$. We then assume $v' \simeq -u'$

- Model for Reynold-Stresses:

$$-\overline{u'v'} = F^2 l^2 \left(\frac{\partial u}{\partial y}\right)^2$$

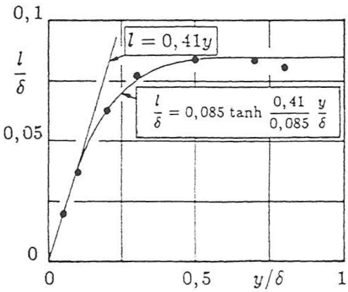
- Experimental Mixing Length:

$$\frac{l}{\delta} = 0.085 \tanh\left(\frac{\chi}{0.085}\frac{y}{\delta}\right)$$

  - $\chi = 0.41$ von Karman constant

- Damping Function:

$$F = 1 - exp\left(-\frac{y^+}{A^+}\right); \;\; A^+ = 26$$



Universal Turbulent BL Profile

Turbulent BL Inner Region

- Near wall hypothesis:
  - $t_a >> t_d$: $\dfrac{\partial \tau}{\partial y} = 0 \longrightarrow \dfrac{\tau}{\tau_w} = 1$
  - $l = \chi y$
- Equations:

$$\tau = \mu \frac{\partial u}{\partial y} + \rho F^2 l^2 \left(\frac{\partial u}{\partial y} \right)^2 = \tau_w$$

$$\frac{\partial u^+}{\partial y^+} + \chi^2 y^{+2} \left[1 - exp\left(-\frac{y^+}{A^+} \right) \right]^2 \left(\frac{\partial u^+}{\partial y^+} \right)^2 = 1$$

- Solutions:
  - Viscous Sublayer ($y^+ < 5$):

$$u^+ = y^+$$

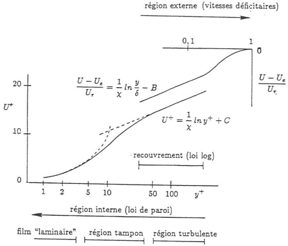
  - Buffer Region ($5 < y^+ < 30$)
  - Log Law ($30 < y^+$):

$$u^+ = \frac{1}{\chi} \ln y^+ + C;\ C \simeq 5.25$$

Turbulent BL Outer Region

- Velocity Defect Law:

$$\frac{u_e - u}{u_\tau} = -\frac{1}{\chi} \ln \frac{y}{\delta} + D$$



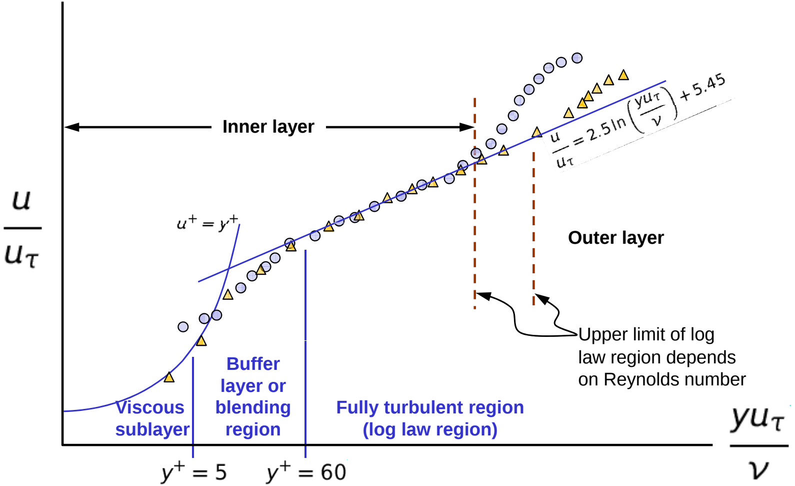
Compatibility:

- Turbulent Friction Law:

$$\left(\frac{C_f}{2} \right)^{-1/2} = \frac{1}{\chi} \ln \left(Re_{\delta_1} \right) + D^*$$

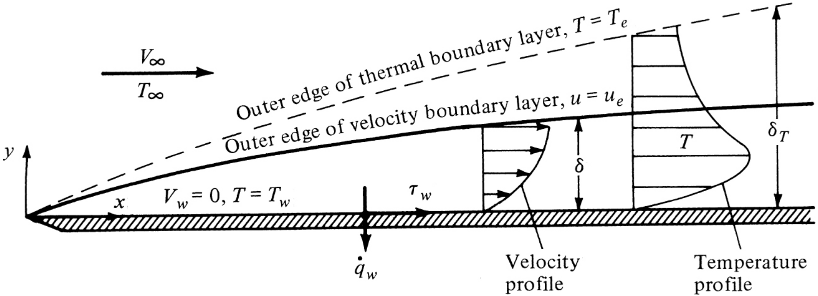
  - $D^* = 4.18$ for flat plate

Universal Turbulent BL Profile



Outline

Momentum & Thermal BL



Two Boundary Layers:

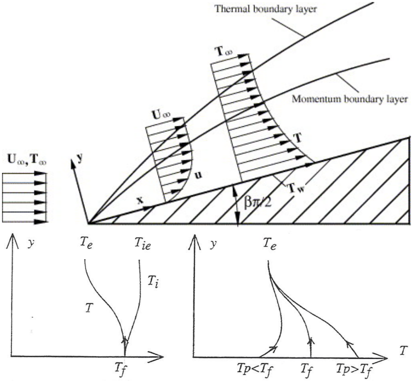
- Momentum BL due to velocity profile
- Thermal BL due to temperature profile
  - Energy-momentum equations coupling due to variable density:
    - Temperature effects (heating/cooling)
    - Compressibility effects at high Mach number

Compressibility & Thermal Effects

Incompressible BL if:

- $M_e = u_e/\sqrt{\gamma r T_e} \lesssim 0.4, 0.5$
- T_w close to T_e

Otherwise compressible



Transonic & Supersonic BL:

- $T_i \simeq$ ct for adiabatic walls
- local M, $T_s \neq$ ct $\longrightarrow \mu$, ρ, λ_c change

Wall properties:

- Adiabatic: $\Phi_w = 0$
  - recovery factor: $r = \frac{h_{aw} - h_e}{h_{ie} - h_e}$
    - laminar: $r \simeq 0.85$
    - turbulent: $r \simeq 0.9$
- Non-adiabatic wall: T_w
  - heat flux coeff: $C_h = \frac{\Phi_w}{\rho_e u_e C_p (T_w - T_{aw})}$
  - Reynolds analogy factor: $s = \frac{C_h}{C_f/2}$
    - flat plate: $s = 1.24$
    - pressure gradient: big changes
- Energy Thickness:

$$\delta_3 = \Delta = \int_0^{\delta} \frac{\rho u}{\rho_e u_e} \left(\frac{h_i}{h_{ie}} - 1 \right) dy$$

2D Compressible BL Equations

- Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$

- y-momentum:

$$\frac{\partial p}{\partial y} = 0$$

- x-momentum:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

- Energy:

$$\rho \frac{\partial h_i}{\partial t} + \rho u \frac{\partial h_i}{\partial x} + \rho v \frac{\partial h_i}{\partial y} = \frac{\partial p}{\partial t} + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y} + \lambda_c \frac{\partial T}{\partial y}\right)$$

- Boundary Conditions:
  - Wall: $u(x, 0; t) = v(x, 0; t) = 0$; $T(x, 0; t) = T_w(x; t)$
  - Edge: $u(x, \infty; t) \longrightarrow u_e(x; t)$; $T(x, \infty; t) \longrightarrow T_e(x; t)$
- Initial Conditions for $t = 0$

2D Compressible BL Integral Equations

Integration with respect to y yields 1-dimensional equations:

$$\int_0^\delta [Eq]\, dy$$

- Continuity:

$$\frac{1}{\rho_e u_e}\frac{d}{dx}\left(\rho_e u_e\left(\delta-\delta_1\right)\right)=\frac{d\delta}{dx}-\frac{v_\delta}{u_e}=C_E$$

- Momentum:

$$\frac{d\theta}{dx}+\theta\left(\frac{H+2}{u_e}\frac{du_e}{dx}+\frac{1}{\rho_e}\frac{d\rho_e}{dx}\right)=\frac{C_f}{2} \qquad \text{with} \qquad \frac{1}{\rho_e}\frac{d\rho_e}{dx}=-\frac{M_e^2}{u_e}\frac{du_e}{dx}$$

- Energy:

$$\frac{1}{\rho_e u_e}\frac{d}{dx}\left(\rho_e u_e \delta_3\right)=\frac{\phi_w}{\rho_e u_e h_{ie}}$$

Flat Plate Compressible BL

- Recovery enthalpy:

$$h_{aw} = h_e \left(1 + r\frac{\gamma - 1}{2} M_e^2\right)$$

  - recovery factor:
    - laminar: $r = Pr^{1/2}$
    - turbulent: $r = Pr^{1/3}$
  - $Pr_{air} = 0.725 \to 0.711$
- Reynolds analogy factor:

$$C_h = s \frac{C_f}{2} \quad \text{with} \quad s = Pr^{-2/3} = 1.24$$

- Wall enthalpy:
  - Adiabatic wall: $h_w = h_{aw}$
  - Prescribed T_w: $h_w = C_p T_w$
- Reference enthalpy:

$$h^* = h_e + 0.54\,(h_w - h_e) + 0.16\,(h_{aw} - h_e)$$

- Wall friction:

$$\frac{C_f}{2} = \frac{a\,f}{Re_x^{s_0}} = \frac{b\,g}{Re_\theta^{m_0}}$$

- Momentum thickness:

$$\frac{\theta}{x} = \frac{k_1\,f}{Re_x^{s_0}} = \frac{k_2\,g}{Re_\theta^{m_0}}$$

- Form Factor:

$$H = H_{inc} + \alpha M_e^2 + \beta \frac{T_w - T_{aw}}{T_e}$$

- Compressible deviations:

$$f = \left(\frac{T_e}{T^*}\right)^{1-s_0} \left(\frac{\mu^*}{\mu_e}\right)^{s_0} \qquad g = f^{m_0+1}$$

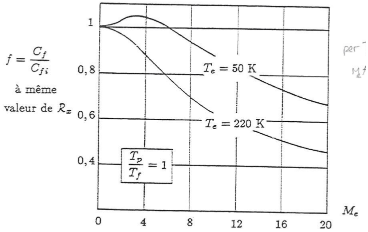
- Sutherland law: $(S = 110.4\,K)$

$$\frac{\mu^*}{\mu_e} = \left(\frac{T^*}{T_e}\right)^{1/2} \frac{1 + S/T_e}{1 + S/T^*}$$

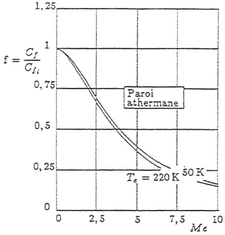
	m_0	s_0	a	b	k_1	k_2	H_{inc}	α	β
laminar	1	1/2	0.332	0.2205	0.664	0.441	2.591	0.667	2.9
turbulent	1/5	1/6	0.0184	0.0086	0.0221	0.0103	1.4	0.4	1.222

Flat Plate Compressible BL Wall Friction

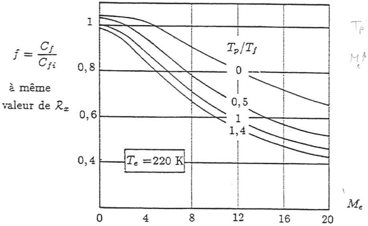
Lam BL, Adiabatic Wall



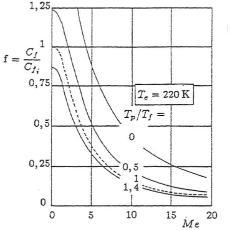
Turb BL, Adiabatic Wall



Lam BL, Conducting Wall



Turb BL, Conducting Wall



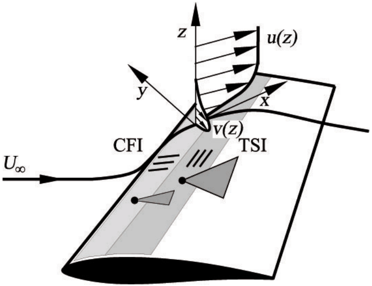
3-Dimensional Boundary Layer

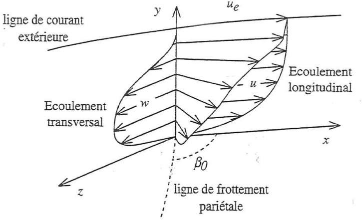
Some examples of 3D-Flows:

- Swept Wing
- Wing-tip vortices
- Fuselage with aoa

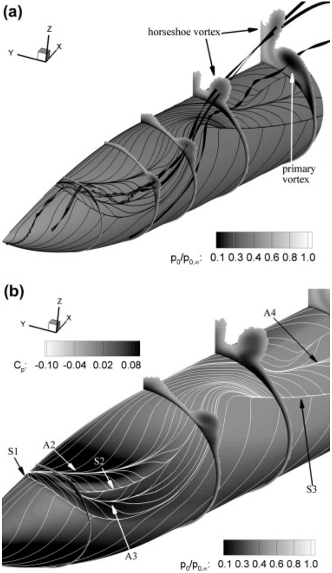
Consequences for BL:

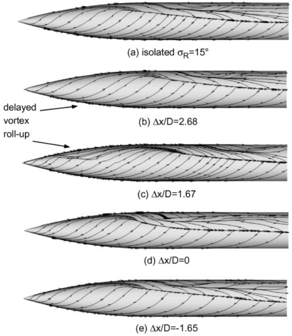
- Secondary flows due to transverse pressure gradients
- Friction lines not parallel to external flow
- Instabilities
- Separation

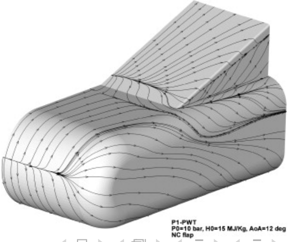




3-Dimensional Boundary Layer







2D Axisymmetric Boundary Layer Equations

For Stationary, 2D Axisymmetric flow:

- Local Equations:

$$
\begin{aligned}
\text{mass:} \quad & \frac{1}{R}\frac{\partial(\rho u R)}{\partial x} + \frac{\partial(\rho v)}{\partial y} &=& \; 0 \\
\text{x-momentum:} \quad & \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &=& \; -\frac{dp}{dx} + \frac{\partial \tau}{\partial y} \\
\text{y-momentum:} \quad & \frac{\partial p}{\partial y} &=& \; 0 \\
\text{energy:} \quad & \rho \frac{\partial h_i}{\partial t} + \rho u \frac{\partial h_i}{\partial x} + \rho v \frac{\partial h_i}{\partial y} &=& \; \frac{\partial p}{\partial t} + \frac{\partial}{\partial y}\left(u\tau - \Phi\right)
\end{aligned}
$$

- With:

$$
\begin{aligned}
\text{state equation:} \quad & \frac{p}{\rho} &=& \; rT \\
\text{stagnation enthalpy:} \quad & h_i &=& \; h + \frac{u^2 + v^2}{2} = h + \frac{u^2}{2} \\
\text{shear stress:} \quad & \tau &=& \; \mu \frac{\partial u}{\partial y} - \rho \, \overline{u' v'} \\
\text{heat flux:} \quad & \Phi &=& \; -\lambda_c \frac{\partial T}{\partial y} + \rho C_p \, \overline{v' T'}
\end{aligned}
$$

2D Axisymmetric Boundary Layer Equations

For Stationary, 2D Axisymmetric flow:

- Integral Equations:

mass:
$$\frac{1}{\rho_e u_e R}\frac{d}{dx}\left(\rho_e u_e R\left(\delta-\delta_1\right)\right) = \frac{d\delta}{dx}-\frac{v_\delta}{u_e}=C_E$$

x-momentum:
$$\frac{d\theta}{dx}+\theta\left(\frac{H+2}{u_e}\frac{du_e}{dx}+\frac{1}{\rho_e}\frac{d\rho_e}{dx}+\frac{1}{R}\frac{dR}{dx}\right) = \frac{C_f}{2}$$

energy:
$$\frac{1}{\rho_e u_e R}\frac{d}{dx}\left(\rho_e u_e R\,\delta_3\right) = \frac{\phi_w}{\rho_e u_e h_{ie}}$$

- With:

isentropic outer flow:
$$\frac{1}{\rho_e}\frac{d\rho_e}{dx} = -\frac{M_e^2}{u_e}\frac{du_e}{dx}$$

Incompressible 2D Axisymmetric Integral Method

- Approximate Solution:

$$\left[\theta R u_e^{(H+2)}\right]_{x_1}^{m_0+1} = \left[\theta R u_e^{(H+2)}\right]_{x_0}^{m_0+1} + (m_0+1)b\int_{x_0}^{x_1}\frac{u_e^{(m_0+1)(H+2)}R^{(m_0+1)}}{(u_e/\nu)^{m_0}}dx$$

Laminar-Turbulent Transition

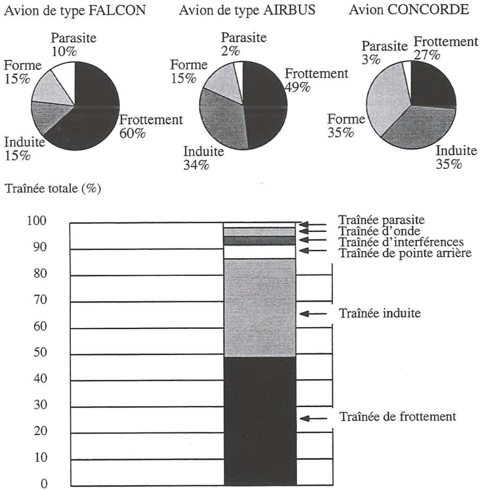
Predicting transition is crucial to drag estimation

- Natural Transition:
  - Orr-Sommerfeld equations (linearised Navier-Stokes)
  - Linear instability: Tollmien-Schlichting waves
  - Secondary instabilities
- Affecting parameters:
  - Adverse pressure gradients
  - Outer flow preturbulent levels and noise
  - Wall-related effects:
    - suction
    - cooling
    - curvature: Görtler vortices
    - rugosity
  - Three-dimensionality:
    - Cross-flow instability (swept wing)
    - Leading edge contamination (fuselage)
- Transition Criteria:
  - Michele's, Granville, e^n Method...

Drag Decomposition

Drag Sources:

- Airfoil drag
  - Parasitic drag
    - Friction drag
    - Form drag
    - Wake drag
    - Interference drag
  - Wave drag
- Lift-induced drag



Outline

Bibliography

1. Schlichting, H; Gersten, K: Boundary-Layer Theory (8th ed.). Springer, Berlin, 2000.
2. Cousteix, J: Modeling and Computation of Boundary-Layer Flows (2nd ed.). Springer, New York, 2005.
3. Anderson, JD: Fundamentals of Aerodynamics (4th ed.). McGraw Hill, New York, 2007.
4. White, FM: Fluid Mechanics. McGraw Hill, New York, 2008.
5. White, FM: Viscous Fluid Flow (3rd ed.). McGraw Hill, New York, 2005.
6. Oertel, H: Prandtl's Essentials of Fluid Mechanics (2nd ed.). Springer, New York, 2004.
7. Bonnet, A; Lunneau, J: Théorie de la Dynamique des Fluides. CEPAD, Toulouse, 1989.
8. Milne-Thomson, LM: Theoretical Aerodynamics (4th ed.). Dover, New York, 1966.
9. Acheson, DJ: Elementary Fluid Dynamics. Oxford University Press, 1998.