

Chapter 2

Limits and continuity of functions of one variable

2.1 Limits

To determine the behavior of a function f as x approaches a finite value c , we use the concept of limit. We say that the limit of f is L , and write $\lim_{x \rightarrow c} f(x) = L$, if the values of f approaches L when x gets closer to c .

Definición 2.1.1. (Limit when x approach a finite value c). We say that $\lim_{x \rightarrow c} f(x) = L$ if for any small positive ϵ , there is a positive δ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - c| < \delta$.

We can split the above definition in two parts, using one-sided limits.

Definición 2.1.2.

1. We say that L is the limit of f as x approaches c from the right, $\lim_{x \rightarrow c^+} f(x) = L$, if for any small positive ϵ , there is a positive δ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < x - c < \delta$.

2. We say that L is the limit of f as x approaches c from the left, $\lim_{x \rightarrow c^-} f(x) = L$, if

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We can also wonder about the behavior of the function f when x approaches $+\infty$ or $-\infty$.

Definición 2.1.4. (Limits when x approaches $\pm\infty$)

1. $\lim_{x \rightarrow +\infty} f(x) = L$ if for any small positive ϵ , there is a positive value of x , call it x_1 , such that

$$|f(x) - L| < \epsilon$$

whenever $x > x_1$.

2. $\lim_{x \rightarrow -\infty} f(x) = L$ if for any small positive ϵ , there is a negative value of x , call it x_1 , such that

$$|f(x) - L| < \epsilon$$

whenever $x < x_1$.

If the absolute values of a function become arbitrarily large as x approaches either a finite value c or $\pm\infty$, then the function has no finite limit L but will approach $-\infty$ or $+\infty$. It is possible to give the formal definitions. For example, we will say that $\lim_{x \rightarrow c} f(x) = +\infty$ if for any large positive number M , there is a positive δ such that

$$f(x) > M$$

whenever $0 < |x - c| < \delta$. Please, complete the remaining cases.

Nota 2.1.5. Note that it could be $c \in D(f)$, so $f(c)$ is well defined, but $\lim_{x \rightarrow c} f(x)$ does not exist or $\lim_{x \rightarrow c} f(x) \neq f(c)$. Consider for instance the function f that is equal to 1 for $x \neq 0$, but $f(0) = 0$. Then clearly the limit of f at 0 is $1 \neq f(0)$.

Ejemplo 2.1.6. Consider the following limits.

1. $\lim_{x \rightarrow 6} x^2 - 2x + 7 = 31$.
2. $\lim_{x \rightarrow \pm\infty} x^2 - 2x + 7 = \infty$, because the leading term in the polynomial gets arbitrarily large.
3. $\lim_{x \rightarrow +\infty} x^3 - x^2 = \infty$, because the leading term in the polynomial gets arbitrarily large for large values of x , but $\lim_{x \rightarrow -\infty} x^3 - x^2 = -\infty$ because the leading term in the polynomial gets arbitrarily large in absolute value, and negative.

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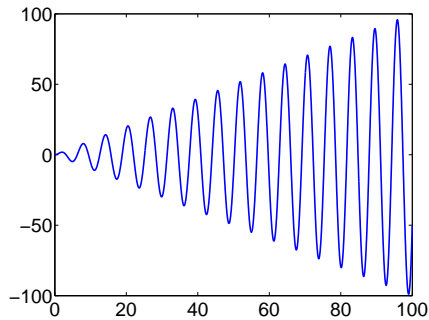
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The right limit is infinity because $1/x$ becomes arbitrarily large when x is small and positive. The left limit is minus infinity because $1/x$ becomes arbitrarily large in absolute value and negative, when x is small and negative.

6. $\lim_{x \rightarrow +\infty} x \sin x$ does not exist. As x approaches infinity, $\sin x$ oscillates between 1 and -1 . This means that $x \sin x$ changes sign infinitely often when x approaches infinity, whilst taking arbitrarily large absolute values. The graph is shown below.



7. Consider the function $f(x) = \begin{cases} x^2, & \text{if } x \leq 0; \\ -x^2, & \text{if } 0 < x \leq 1; \\ x, & \text{if } x > 1. \end{cases}$ $\lim_{x \rightarrow 0} f(x) = f(0) = 0$, but $\lim_{x \rightarrow 1} f(x)$ does not exist since the one-sided limits are different.

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x = 1, \\ \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} -x^2 = -1. \end{aligned}$$

8. $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist, because the one-sided limits are different.

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{|x|}{x} &= \lim_{x \rightarrow 0^+} \frac{x}{x} = 1, \\ \lim_{x \rightarrow 0^-} \frac{|x|}{x} &= \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1 \quad (\text{when } x \text{ is negative, } |x| = -x). \end{aligned}$$

In the following, $\lim f(x)$ refer to the limit as x approaches $+\infty$, $-\infty$ or a real number

but we never miss different types of limits.

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3. *Product*: $\lim f(x)g(x) = (\lim f(x))(\lim g(x))$.

4. *Quotient*: If $\lim g(x) \neq 0$, then $\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$.

Teorema 2.1.7 (Squeeze Theorem). Assume that the functions f , g and h are defined around the point c , except, maybe, for the point c itself, and satisfy the inequalities

$$g(x) \leq f(x) \leq h(x).$$

Let $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$. Then

$$\lim_{x \rightarrow c} f(x) = L.$$

Ejemplo 2.1.8. Show that $\lim_{x \rightarrow 0} x \operatorname{sen} \left(\frac{1}{x} \right) = 0$.

SOLUTION: We use the theorem above with $g(x) = -|x|$ and $h(x) = |x|$. Notice that for every $x \neq 0$, $-1 \leq \operatorname{sen}(1/x) \leq 1$ thus, when $x > 0$

$$-x \leq x \operatorname{sen}(1/x) \leq x,$$

and when $x < 0$

$$x \leq x \operatorname{sen}(1/x) \leq -x.$$

These inequalities mean that $-|x| \leq x \operatorname{sen}(1/x) \leq |x|$. Since

$$\lim_{x \rightarrow 0} -|x| = \lim_{x \rightarrow 0} |x| = 0,$$

we can use the theorem above to conclude that $\lim_{x \rightarrow 0} x \operatorname{sen} \frac{1}{x} = 0$.

2.1.2 Techniques for evaluating $\lim \frac{f(x)}{g(x)}$

1. Use the property of the quotient of limits, if possible.
2. If $\lim f(x) = 0$ and $\lim g(x) = 0$, try the following:
 - (a) Factor $f(x)$ and $g(x)$ and reduce $\frac{f(x)}{g(x)}$ to lowest terms.
 - (b) If $f(x)$ or $g(x)$ involves a square root, then multiply both $f(x)$ and $g(x)$ by the

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- If $f(x) \neq 0$ and $\lim g(x) = 0$, then either $\lim \frac{f(x)}{g(x)}$ does not exist or $\lim \frac{f(x)}{g(x)} = +\infty$ or $-\infty$.
- If x approaches $+\infty$ or $-\infty$, divide the numerator and denominator by the highest power of x in any term of the denominator.

Ejemplo 2.1.10.

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2x}{-x^4 + 2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{2}{x^3}}{-1 + \frac{2}{x^4}} = \frac{0 - 0}{-1 + 0} = 0.$$

2.1.3 Exponential limits

Let the limit

$$\lim_{x \rightarrow c} [f(x)]^{g(x)}$$

be an indetermination. This happens if

- $\lim_{x \rightarrow c} f(x) = 1$ and $\lim_{x \rightarrow c} g(x) = \infty$ (1^∞).
- $\lim_{x \rightarrow c} f(x) = 0$ and $\lim_{x \rightarrow c} g(x) = 0$ (0^0).
- $\lim_{x \rightarrow c} f(x) = \infty$ and $\lim_{x \rightarrow c} g(x) = 0$ (∞^0).

Noting that

$$\lim_{x \rightarrow c} [f(x)]^{g(x)} = \lim_{x \rightarrow c} e^{g(x) \ln f(x)} = e^{\lim_{x \rightarrow c} g(x) \ln f(x)},$$

all cases are reduced to the indetermination $0 \cdot \infty$, since we have to compute the limit

$$\lim_{x \rightarrow c} g(x) \ln f(x).$$

In the first indetermination, 1^∞ , it often helps to use the identity

$$\lim_{x \rightarrow c} g(x) \ln f(x) = \lim_{x \rightarrow c} g(x)(f(x) - 1).$$

since when x is close to 0, $\ln(1+x) \approx x$, or, $\ln x \approx x-1$

Ejemplo 2.1.11. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} e^{x \ln \left(1 + \frac{1}{x}\right)} = e^{x \frac{1}{x}} = e.$

Ejemplo 2.1.12. Let $a, b > 0$. Calculate $\lim_{x \rightarrow \infty} \left(\frac{1+ax}{2+bx}\right)^x$.

If $a > b$, then the basis function tends to $a/b > 1$, thus the limit is ∞ . If $a < b$, then the basis function tends to $a/b < 1$, thus the limit is 0. When $a = b$

$$\lim_{x \rightarrow \infty} \left(\frac{1+ax}{2+bx}\right)^x = e^{\lim_{x \rightarrow \infty} x \left(\frac{1+ax}{2+bx} - 1\right)} = e^{\lim_{x \rightarrow \infty} \frac{-x}{2+bx}} = e^{-1/a}.$$

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$\lim_{x \rightarrow 0} \frac{\cos x}{x} = \lim_{x \rightarrow 0} \frac{-\sin x}{1} = 0$

$$2. \lim_{x \rightarrow 0} \frac{\text{sen } 3x}{x} \stackrel{\{z=3x\}}{=} \lim_{z \rightarrow 0} \frac{\text{sen } z}{\frac{z}{3}} = 3 \lim_{z \rightarrow 0} \frac{\text{sen } z}{z} = 3.$$

2.2 Asymptotes

An *asymptote* is a line that the graph of a function approaches more and more closely until the distance between the curve and the line almost vanishes.

Definición 2.2.1. Let f be a function

1. The line $x = c$ is a vertical asymptote of f if $\lim_{x \rightarrow c^+} |f(x)| = \infty$ or $\lim_{x \rightarrow c^-} |f(x)| = \infty$.
2. The line $y = b$ is a horizontal asymptote of f if $\lim_{x \rightarrow +\infty} f(x) = b$ or $\lim_{x \rightarrow -\infty} f(x) = b$.
3. The line $y = ax + b$ is an oblique asymptote of f if
 - (a) $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = a$ and $\lim_{x \rightarrow +\infty} (f(x) - ax) = b$, or
 - (b) $\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = a$ and $\lim_{x \rightarrow -\infty} (f(x) - ax) = b$.

Notice that a horizontal asymptote is a particular case of oblique asymptote with $a = 0$.

Ejemplo 2.2.2. Determine the asymptotes of $f(x) = \frac{(1+x)^4}{(1-x)^4}$.

SOLUTION: Since the denominator vanishes at $x = 1$, the domain of f is $\mathbb{R} - \{1\}$. Let us check that $x = 1$ is a vertical asymptote of f :

$$\lim_{x \rightarrow 1^\pm} \frac{(1+x)^4}{(1-x)^4} = +\infty$$

On the other hand

$$\lim_{x \rightarrow +\infty} \frac{(1+x)^4}{(1-x)^4} = \lim_{x \rightarrow +\infty} \frac{(1/x+1)^4}{(1/x-1)^4} = 1$$

hence $y = 1$ is a horizontal asymptote at $+\infty$. In the same way, $y = 1$ is a horizontal asymptote at $-\infty$. There is no other oblique asymptotes.

Ejemplo 2.2.3. Determine the asymptotes of $f(x) = \frac{3x^3 - 2}{x^2}$.

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$x \rightarrow \pm\infty$ $x =$ $x \rightarrow \pm\infty$ $x =$

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thus, there is no horizontal asymptote. Let us study now oblique asymptotes:

$$a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{3x^3 - 2}{x^3} = \lim_{x \rightarrow \pm\infty} \left(3 - \frac{2}{x^3} \right) = 3,$$
$$b = \lim_{x \rightarrow \pm\infty} (f(x) - 3x) = \lim_{x \rightarrow \pm\infty} \left(\frac{3x^3 - 2}{x^2} - 3x \right) = \lim_{x \rightarrow \pm\infty} \left(-\frac{2}{x^2} \right) = 0.$$

We conclude that $y = 3x$ is an oblique asymptote both at $+\infty$ and $-\infty$.

2.3 Continuity

The easiest limits to evaluate are those involving continuous functions. Intuitively, a function is continuous if one can draw its graph without lifting the pencil from the paper.

Definición 2.3.1. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at c if $c \in D(f)$ and

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Hence, f is *discontinuous at c* if either $f(c)$ is undefined or $\lim_{x \rightarrow c} f(x)$ does not exist or $\lim_{x \rightarrow c} f(x) \neq f(c)$.

2.3.1 Properties of continuous functions

Suppose that the functions f and g are both continuous at c . Then the following functions are also continuous at c .

1. *Sum.* $f + g$.
2. *Product by a scalar.* λf , $\lambda \in \mathbb{R}$.
3. *Product.* fg .
4. *Quotient.* f/g , whenever $g(c) \neq 0$.

2.3.2 Continuity of a composite function

Suppose that f is continuous at c and g is continuous at $f(c)$. Then, the composite function $g \circ f$ is also continuous at c .

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1. The function $f(x) = \sqrt{4-x^2}$ is the composition of the functions $y = 4-x^2$ and $f(y) = y^{1/2}$, which are elementary, thus f is continuous in its domain, that is, in $D = [-2, +2]$.
2. The function $g(x) = \frac{1}{\sqrt{4-x^2}}$ is the composition of function f above and function $g(y) = 1/y$, thus it is elementary and continuous in its domain, $D(g) = (-2, +2)$.

2.3.4 Limit of a composite function

Let f, g be functions from \mathbb{R} to \mathbb{R} and $c \in \mathbb{R}$. If g is continuous at L and $\lim_{x \rightarrow c} f(x) = L$, then

$$\lim_{x \rightarrow c} g(f(x)) = g(\lim_{x \rightarrow c} f(x)) = g(L).$$

Ejemplo 2.3.3. Show that $\lim_{x \rightarrow 1} \arctan\left(\frac{x^2 + x - 2}{3x^2 - 3x}\right) = \frac{\pi}{4}$.

SOLUTION: The function \tan^{-1} is continuous.

$$\begin{aligned} \lim_{x \rightarrow 1} \arctan\left(\frac{x^2 + x - 2}{3x^2 - 3x}\right) &= \arctan\left(\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{3x^2 - 3x}\right) \\ &= \arctan\left(\lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{3x(x-1)}\right) \\ &= \arctan\left(\lim_{x \rightarrow 1} \frac{x+2}{3x}\right) \\ &= \arctan 1 \\ &= \frac{\pi}{4}. \end{aligned}$$

Ejemplo 2.3.4. Evaluate the following limits:

- $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \ln(1+x)^{1/x} = \ln\left(\lim_{x \rightarrow 0} (1+x)^{1/x}\right) = \ln e = 1.$

Notice that the function $\ln(\cdot)$ is continuous at e , thus we can apply 2.3.4.

- $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \lim_{z \rightarrow 0} \frac{z}{\frac{\ln(1+z)}{\ln a}} = \ln a \left(\lim_{z \rightarrow 0} \frac{z}{\ln(1+z)}\right) = \ln a.$



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Ejemplo 2.3.6. Show that the equation $x^3 + x - 1 = 0$ admits a solution, and find it with an error less than 0.1.

SOLUTION: With $f(x) = x^3 + x - 1$ the problem is to show that there exists c such that $f(c) = 0$. We want to apply Bolzano's Theorem. First, f is continuous in \mathbb{R} . Second, we identify a suitable interval $I = [a, b]$. Notice that $f(0) = -1 < 0$ and $f(1) = 1 > 0$ thus, there is a solution $c \in (0, 1)$.

Now, to find an approximate value for c , we use a method of *interval-halving* as follows: consider the interval $[0.5, 1]$; $f(0.5) = 1/8 + 1/2 - 1 < 0$ and $f(1) > 0$, thus $c \in (0.5, 1)$. Choose now the interval $[0.5, 0.75]$; $f(0.5) < 0$ and $f(0.75) = 27/64 + 3/4 - 1 > 0$ thus, $c \in (0.5, 0.75)$. Let now the interval $[0.625, 0.75]$; $f(0.625) \approx -0.13$ and $f(0.75) > 0$ thus, $c \in (0.625, 0.75)$. The solution is approximately $c = 0.6875$ with a maximum error of 0.0625.

Teorema 2.3.7 (Weierstrass' Theorem). *If f is continuous in $[a, b]$, then there exist points $c, d \in [a, b]$ such that*

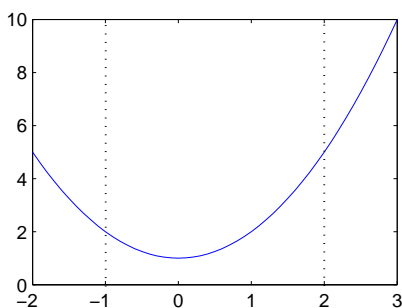
$$f(c) \leq f(x) \leq f(d)$$

for every $x \in [a, b]$.

The theorem asserts that a continuous function attains over a closed interval a minimum ($m = f(c)$) and a maximum value ($M = f(d)$). The point c is called a *global minimizer* of f on $[a, b]$ and d is called a *global maximizer* of f on $[a, b]$.

Ejemplo 2.3.8. Show that the function $f(x) = x^2 + 1$ attains over the closed interval $[-1, 2]$ a minimum and a maximum value.

SOLUTION: The graph of f is shown below.



We can see that f is continuous in $[-1, 2]$, actually f is continuous in \mathbb{R} , and f attains

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- Take $I = [0, \infty)$ and $f(x) = 1/(1+x)$; f is continuous in I , but it does not have global minimum, since $\lim_{x \rightarrow \infty} f(x) = 0$, but $f(x) > 0$ is strictly positive for every $x \in I$.
- The function is not continuous. Take $I = [0, 1]$ and $f(x) = \begin{cases} x, & \text{if } 0 \leq x < 1; \\ 0, & \text{if } x = 1. \end{cases}$; f has a global minimum at $x = 0$, but there is no global maximum since $\lim_{x \rightarrow 1} f(x) = 1$ but $f(x) < 1$ for every $x \in I$.



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