Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 1

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Outline

1. Introduction and Basic Notions
    - Definition of Vectorial Space
    - Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$
    - Standard basis and parametric equations
    - Examples

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Definition

A **set** *is a collection of defined objects.* . In other words a set is given by his elements and has a property that determines whatever is or not in the set.

Examples,

- The set of natural numbers less than 3.
- The set of the prime numbers. ç

We denote $\mathbf{x} \in \mathbf{S}$ when a element x belongs to the set S. We can describe a set by showing explicitly his elements or we can describe the set using a condition it satisfies:

$S = \{x : x \text{ satisfies condition } P\}$. For example,
$A = \{x : x \in \mathbb{N} \text{ and } x < 3\}$.

Definition

A **correspondence** *is any rule who associate elements of a set A with elements of a set B.*

Definition

An **application** *or* **function** *is a correspondence where any element of A is associated with an element of B and only one.*

We call **image** or **range** to the set
f(A)=$\{b \in B$: where exists a, such as $f(a) = b\}$.
An application can be:

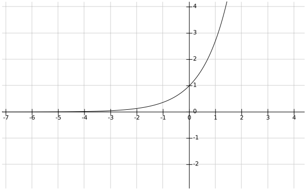
- **Injective**. If different elements has different images. So $f(a) = f(a')$ means $a = a'$.
- **Surjective**. If $f(A) = B$. That is that $\forall b \in B$, exists $a \in A$ such as $b = f(a)$.
- **Bijective**. If it is injective and Surjective.

Examples:

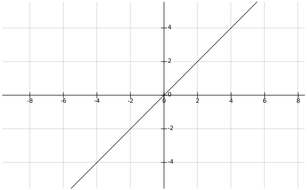
- $f(x) = x^2$, $x \in \mathbb{R}$ is just an application.But if we define x only in $\mathbb{N}$ is injective and if we define x in $\mathbb{R}^+$ is bijective.
- $g(x) = e^x$, is injective but not bijective.
- $h(x) = x$, is bijective.

As you see the notion doesn't depend only on the function f but also of the set where it is defined. If an application is bijective it has **inverse** $f^{-1} : B \to A$ and is also bijective.

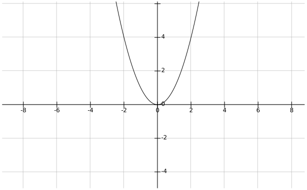
Given two functions $f : A \to B$ and $g : C \to D$, with $f(A) \subseteq C$ his **composition**, $g \circ f$ is an app from A $\to$ D defined as $g(f(a))$.
It is easily verified that in general $g \circ f \neq f \circ g$ but is associative. Try for example: $f(x) = x^2$ and $g(x) = x/\sqrt{x^2 + 1}$.



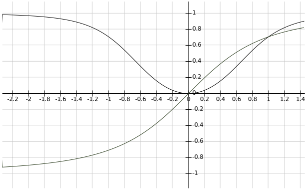
(a) $y = e^x$



(b) $y = x$



(c) $y = x^2$



(d) Composition: $f \circ g$ and $g \circ f$

Figura: Examples of functions.

To define a **Vectorial Space** mathematically, we would need to introduce the notion of *group*, *ring*, *field* and more. It is not the goal of this course.

It would look like,
Let be $\mathbb{K}$ a field and V a non empty set, then V is a vectorial space over $\mathbb{K}$ if...

It is not very tempting right?

The branches of maths that studies this are called *linear algebra* and *algebraic structures*. As a curios comment, for mathematician a vectorial space is more than our geometrically intuitive space $\mathbb{R}^2$ or $\mathbb{R}^3$. Mathematically, the space of all matrices $\mathbb{M}_{m_x n}$ or the space of the group of all real function are also Vectorial Spaces.

Definition

For us a vectorial space will be $\mathbb{R}^n$ for $n = 1, 2, 3, ...$ associated with two operations defined over the set $\mathbb{R}^n$: the **sum** and the **scalar multiplication**.

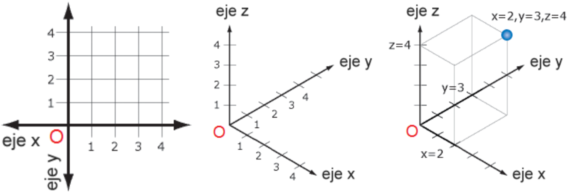


Figura: $\mathbb{R}^2$ and $\mathbb{R}^3$.

Outline

Notation

- We will use **boldface** letters to denote vectors

$$\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2 \quad \text{or} \quad \mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$$

- We will refer to single real numbers as **scalars**, $a \in \mathbb{R}$.

Definition

- A **vector** in $\mathbb{R}^2$ is an ordered pair of real numbers

$$(a_1, a_2) \in \mathbb{R}^2, \quad \text{e.g.,} \quad (\pi, 17) \in \mathbb{R}^2$$

- A **vector** in $\mathbb{R}^3$ is an ordered triple of real numbers

$$(a_1, a_2, a_3) \in \mathbb{R}^3, \quad \text{e.g.,} \quad (\pi, e, \sqrt{2}) \in \mathbb{R}^3$$

Algebraic and Geometric Perspectives

- The Notions of a **vector** is fundamental for calculus of several variables.
- There are always two points of view: **algebraic** (above definition) and **geometric** (visual interpretation).
- Both perspectives are necessary in order to solve problems effectively.

Definition

- Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ are **equal** if their corresponding components are equal

$$a_1 = b_1$$
$$a_2 = b_2$$
$$a_3 = b_3$$

- The same definition holds for vectors in $\mathbb{R}^2$.

Example

- Vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (\frac{3}{3}, \frac{6}{3})$ are equal in $\mathbb{R}^2$.
- Vectors $\mathbf{c} = (1, 2, 3)$ and $\mathbf{d} = (2, 3, 1)$ are not equal in $\mathbb{R}^3$.

Definition: Vector Addition

- Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbb{R}^3$
- The **vector sum** $\mathbf{a} + \mathbf{b}$ is the vector in $\mathbb{R}^3$ obtained via componentwise addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Example

- $(0, 1, 3) + (7, -2, 10) = (7, -1, 13)$ in $\mathbb{R}^3$
- $(1, 1) + (\pi, \sqrt{2}) = (1 + \pi, 1 + \sqrt{2})$ in $\mathbb{R}^2$

Sum properties:

Properties of Vector Addition

1. **Commutativity**: $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ for all $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$
2. **Associativity**: $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$
3. **Zero Vector** or neutral element: a special vector $\mathbf{0}=(0,0,0)$ with the property that $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in\mathbb{R}^3$.

Now we define the scalar multiplication:

Scalar Multiplication

- Let $\mathbf{a} = (a_1, a_2, a_3)$ be a vector in $\mathbb{R}^3$.
- Let $k \in \mathbb{R}$ be a scalar (real number).
- The **scalar product** $k\mathbf{a}$ is the vector in $\mathbb{R}^3$ given by multiplying each component of $\mathbf{a}$ by k

$$k\mathbf{a} = (ka_1, ka_2, ka_3)$$

Example

- If $\mathbf{a} = (2, 0, \sqrt{2})$ and $k = 7$ then $k\mathbf{a} = (14, 0, 7\sqrt{2})$.

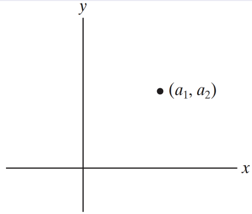
Properties of Scalar Multiplication

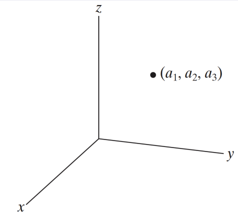
For all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ and scalars k and l in $\mathbb{R}$, we have

1. $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$ (**distributivity**)
2. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ (**distributivity**)
3. $k(l\mathbf{a}) = (kl)\mathbf{a} = l(k\mathbf{a})$

First Interpretation: Vectors as points

- A vector **a** in $\mathbb{R}^2$ may be thought of as a **point** in plane $\mathbb{R}^2$ and a vector **a** in $\mathbb{R}^3$ may be thought of as a **point** in space $\mathbb{R}^3$:

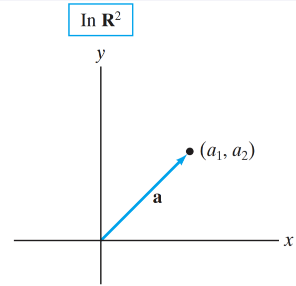


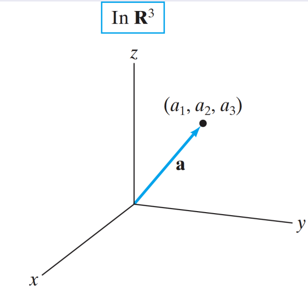


- This interpretation in terms of points has not meaningful geometric interpretation.

Second Interpretation: Vectors as Positions

- We can visualise a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ as an **arrow** that begins at the origin and ends at the point.
  We associate a vector with the point where it ends (bijective application). In this way the elements of $\mathbb{R}^n$ are points but also vectors.
- Such a description is often referred to as the **position vector** of the point (a_1, a_2) or (a_1, a_2, a_3).





As we usually have been told, vectors have **magnitude** and **direction**:

Second Interpretation: Vectors as Positions

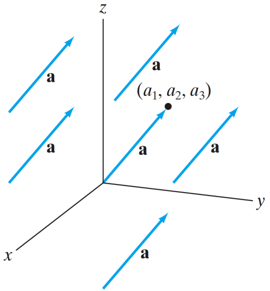
- We take **magnitude** to mean **length of the arrow**.
- We take **direction** to be the **orientation or sense of the arrow**.

Note

- There is an exception to this approach, the **zero vector**.
- It just sits at the origin, like a point.
- It has no magnitude and, therefore, an indeterminate direction.

Second Interpretation: Vectors as Positions

- In physics, not all vectors are represented by arrows having their tails bound to the origin.
- We need "the freedom" to **parallel translate** vectors throughout $\mathbb{R}^2$ and $\mathbb{R}^3$.
- One may represent the vector $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at any point.



For this we need the following definition:

Definition

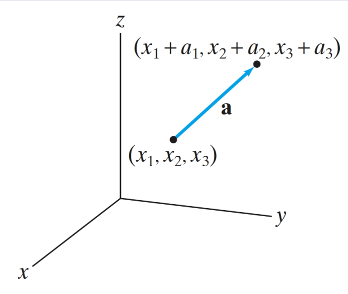
A **free vector** *is a vector with an start not necessarily at the origin. Two of this vectors are the same if we can obtain one from another with a movement of translation*.

Note

The previous intuitive definition of free vectors, can be mathematically defined as a so-called **affine space**. Not necessary for this course.

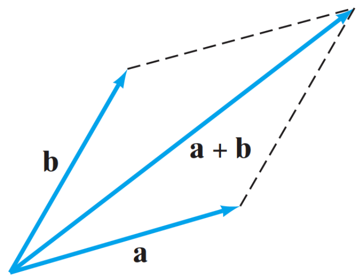
Second Interpretation: Vectors as Positions

- If we wish to represent $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at the point
$$(x_1, x_2, x_3)$$
- Then, the head of the arrow would be at the point
$$(x_1 + a_1, x_2 + a_2, x_3 + a_3)$$



Vector Addition: geometric interpretation

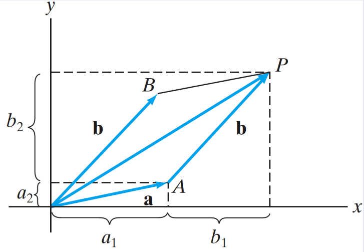
- It is the so-called **parallelogram law**.
- Assume $\mathbf{a}$ and $\mathbf{b}$ are nonparallel vectors drawn with their tails emanating from the same point.
- Then $\mathbf{a} + \mathbf{b}$ may be represented by the arrow that runs along a diagonal of the parallelogram.



Vector Addition: algebraic and geometric

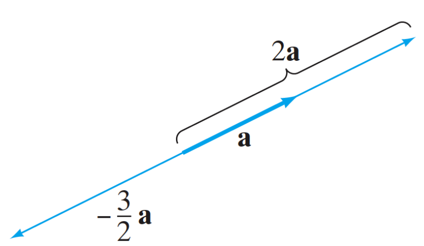
- We check that geometric constructions agree with algebraic definitions.
- Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be two vectors in $\mathbb{R}^2$.
- The arrow obtained from the parallelogram law addition is the one whose tail is at the origin and whose head is at the point:

$$P = (a_1 + b_1, a_2 + b_2)$$



Scalar multiplication: algebraic vs geometric

- Scalar multiplication is easier to visualise.
- The vector $k\mathbf{a}$ may be represented. by an arrow whose:
  - length is $|k|$ times the length of $\mathbf{a}$.
  - direction is the same as that of $\mathbf{a}$ when $k > 0$ and the opposite when $k < 0$.

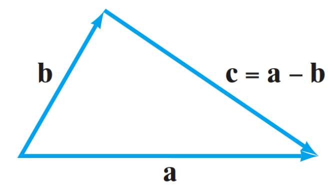


Vector subtraction: algebraic Notions and geometric visualization

- The **difference** $\mathbf{a} - \mathbf{b}$ between two vectors is defined as

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

- It may be represented by an arrow pointing from the head of $\mathbf{b}$ toward the head of $\mathbf{a}$

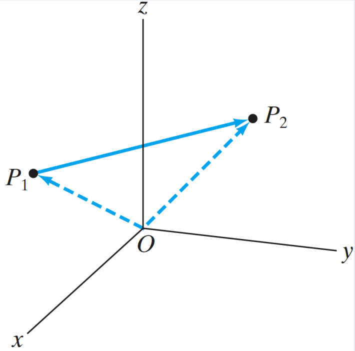


- Such an arrow is also a diagonal of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Definition 1.5: the displacement vector

- Given two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in $\mathbb{R}^3$, the **displacement vector** from P_1 to P_2 is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$



Example: position and velocity of a particle

- Suppose a particle in space is at the point (a_1, a_2, a_3).
- Then, the particle has position vector

$$\mathbf{a} = (a_1, a_2, a_3)$$

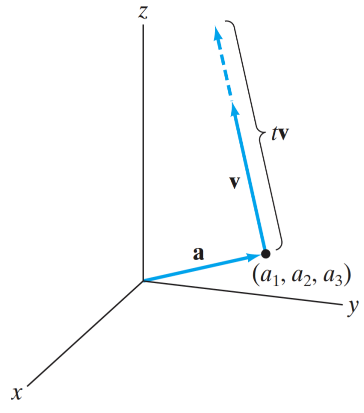
- Assume that the particle travels with constant velocity $\mathbf{v} = (v_1, v_2, v_3)$ for t seconds

Which is the particle's displacement from its original position?

Which is its new coordinate position?

Example: position and velocity of a particle

- The particle's displacement from its original position is $t\mathbf{v}$
- Its new coordinate position is $\mathbf{a} + t\mathbf{v}$



Example 5

- The S.S. Calculus is cruising due south at a rate of 15 knots (nautical miles per hour) with respect to still water.
- However, there is also a current of $5\sqrt{2}$ knots southeast.

What is the total velocity of the ship?

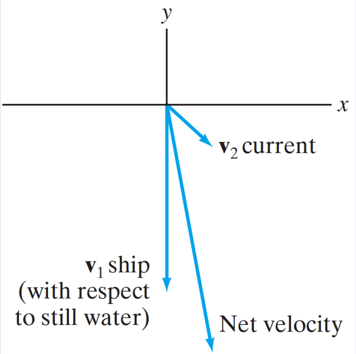
If the ship is initially at the origin and a lobster pot is at position $(20, -79)$, will the ship collide with the lobster pot?

Example 5

- Since velocities are vectors, the total velocity of the ship is

$$\mathbf{v_1} + \mathbf{v_2}$$

  - $\mathbf{v_1}$ is the velocity of the ship with respect to still water
  - $\mathbf{v_2}$ is the southeast-pointing velocity of the current



Example 5

- We easily know that $\mathbf{v_1} = (0, -15)$
- Since $\mathbf{v_2}$ points southeastward, its direction must be along the line
$$y = -x$$
- Therefore, $\mathbf{v_2}$ can be written as $\mathbf{v_2} = (v, -v)$, where v is a positive real number
- By the Pythagorean theorem, if the length of $\mathbf{v_2}$ is $5\sqrt{2}$, then
$$v^2 + (-v)^2 = (5\sqrt{2})^2 \Rightarrow 2v^2 = 50 \Rightarrow v = 5 \Rightarrow \mathbf{v_2} = (5, -5)$$
- Hence, the net velocity is
$$(0, -15) + (5, -5) = (5, -20)$$

Example 5

- After 4 hours, therefore, the ship will be at position

$$(0,0)+4(5,-20)=(20,-80)$$

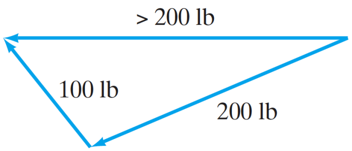
- Thus, it will miss the lobster pot.

Example 6

- The theory behind the art of judo is an excellent example of vector addition.
- If two people, one relatively strong and the other relatively weak, have a shoving match, it is clear who will prevail.
- Someone pushing one way with 200 lb of force will succeed in overpowering another pushing the oppositeway with 100 lb.
- Indeed, the net force will be 100 lb in the direction in which the stronger person is pushing.

Example

- The weaker participant applies his or her 100 lb of force in a direction only slightly different from that of the stronger.
- He or she will effect a vector sum of length large enough to surprise the opponent.



- This is the basis for essentially all of the throws of judo.
- This is why judo is described as:

The art of using a person's strength against himself or herself

Outline

1. Introduction and Basic Notions
   - Definition of Vectorial Space
   - Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$
   - Standard basis and parametric equations
   - Examples

The Standard Basis Vectors in $\mathbb{R}^2$

- In $\mathbb{R}^2$, a special notational role is played by the vectors

$$e_1 = \mathbf{i} = (1,0) \text{ and } e_2 = \mathbf{j} = (0,1)$$

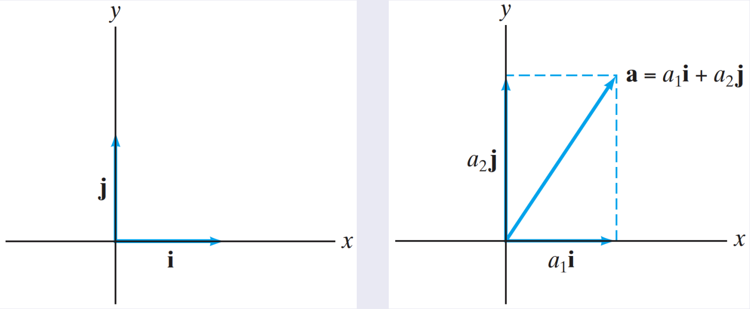
- In mathematics is more common to use $\mathbf{e_1}, \mathbf{e_2}$ notation and in engineering the $\mathbf{i}$, $\mathbf{j}$ notation. These vector form what is called the standard or **canonical base**.
- They form a basis because any vector $\mathbf{a} = (a_1, a_2)$ may be written in terms of them via vector addition and scalar multiplication:

$$(a_1, a_2) = (a_1, 0) + (0, a_2) = a_1(1,0) + a_2(0,1) = a_1\mathbf{i} + a_2\mathbf{j}$$

- They are called canonical because is the natural and most common basis. Nevertheless, any two **linear independent** vectors can be a base.

The Standard Basis Vectors in $\mathbb{R}^2$

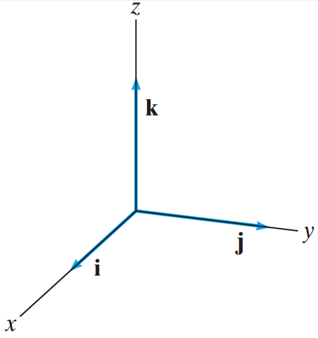
- Geometrically, there is a straightforward significance of the **standard basis vectors i** and **j**.
- An arbitrary vector **a** $\in \mathbb{R}^2$ can be decomposed into appropriate vector components along the x- and y-axes.



The Standard Basis Vectors in $\mathbb{R}^3$

- Analogously, the standard basis in $\mathbb{R}^3$ is

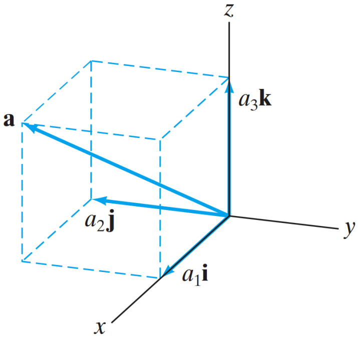
$$\mathbf{i} = (1, 0, 0), \ \mathbf{j} = (0, 1, 0) \text{ and } \mathbf{k} = (0, 0, 1)$$



The Standard Basis Vectors in $\mathbb{R}^3$

- Any vector $\mathbf{a} = (a_1, a_2, a_3)$ may also be written as

$$a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$



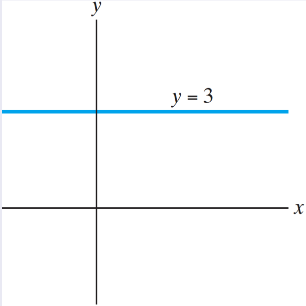
Straight lines in $\mathbb{R}^2$

- In $\mathbb{R}^2$, straight lines are described by equations of the form

$$y = mx + b$$

or

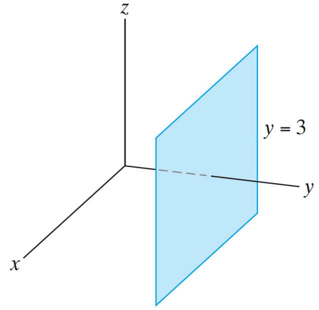
$$Ax + By = C$$



Straight lines in $\mathbb{R}^3$

- One might expect the same sort of equation for a line in $\mathbb{R}^3$

However, a single such linear equation
describes a plane, not a line



- A pair of simultaneous equations in x, y and z is required to define a line.

Parametric equations of a curve in $\mathbb{R}^2$

- A curve in the plane may be described analytically by points (x, y) where
  - x and y are given as functions of a third independent variable. t
  - variable t is known as the **parameter**.
- These functions give rise to **parametric equations** for the curve.

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

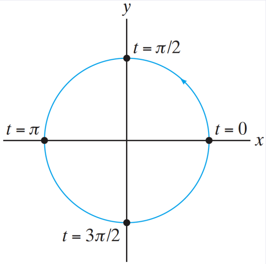
Example

- Consider the set of equations

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0 \leq t < 2\pi$$

- They describe a circle of radius 2, since we may check that

$$x^2 + y^2 = (2\cos t)^2 + (2\sin t^2) = 4 = 2^2$$



Parametric equations of a curve in $\mathbb{R}^3$

- Parametric equations may be used as readily to describe curves in $\mathbb{R}^3$.
- A curve in $\mathbb{R}^3$ is the set of points (x, y, z) whose coordinates x, y and z are each given by a function of t,

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$

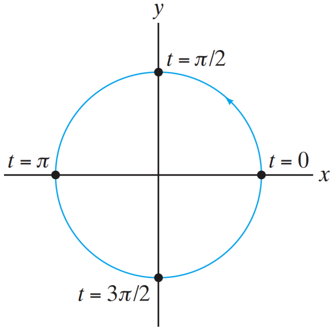
Parametric equations: Advantages

- The advantages of using parametric equations are twofold:
  - First, they offer a uniform way of describing curves in any number of dimensions.
  - Second, they allow you to get a dynamic sense of a curve.

Consider the parameter variable t to represent time and imagine that a particle is travelling along the curve with time

Parametric equations: geometric visualization

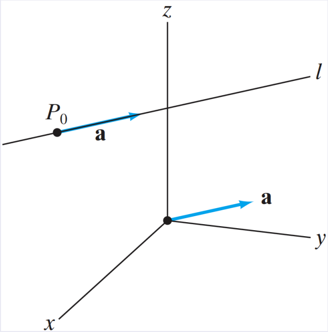
- Geometrically we can assign a **direction** to the curve to signify increasing t
- Notice the arrow:



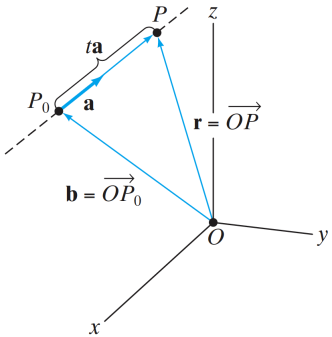
Parametric equations of lines in $\mathbb{R}^n$

As we know from high-school, a line in $\mathbb{R}^2$ or $\mathbb{R}^3$ is uniquely determined by two pieces of geometric information:

1. A vector whose direction is parallel to that of the line.
2. Any particular point lying on the line.



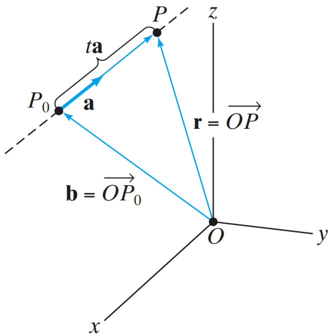
Parametric equations of lines in $\mathbb{R}^n$



- We call the vector

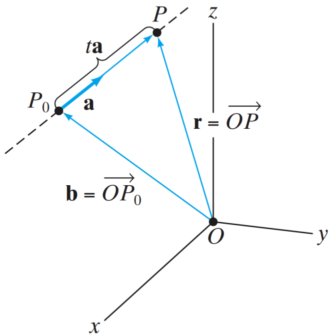
$$\mathbf{r} = \overrightarrow{OP}\ \textbf{ the position vector } \text{of } P(x, y, z).$$

Parametric equations of lines in $\mathbb{R}^n$



- $\mathbf{r} = \overrightarrow{OP}$ can be seen as sum of:
  - The position vector $\mathbf{b}$ of the point P_0 (i.e., $\overrightarrow{OP_0}$), and
  - A vector parallel to $\mathbf{a}$.

Parametric equations of lines in $\mathbb{R}^n$



- Any vector parallel to **a** must be a scalar multiple of **a**.
- Letting this scalar be the parameter variable t, we have

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OP_0} + t\mathbf{a}$$

Proposition

The vector parametric equation for the line through the point $P_0 = (b_1, b_2, b_3)$, whose position vector is

$$\overrightarrow{OP_0} = \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

and parallel to

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

is:

$$\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}$$

Proposition

- Expanding formula $\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}$

$$\begin{aligned} \mathbf{r}(t) &= \overrightarrow{OP} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} + t(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \\ &= (a_1 t + b_1)\mathbf{i} + (a_2 t + b_2)\mathbf{j} + (a_3 t + b_3)\mathbf{k} \end{aligned}$$

- Let P has coordinates (x, y, z)

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- Thus, our parametric equations are

$$\begin{cases} x = a_1 t + b_1 \\ y = a_2 t + b_2 \qquad t \in \mathbb{R} \\ z = a_3 t + b_3 \end{cases}$$

Proposition

- These parametric equations work just as well in $\mathbb{R}^n$
- We take $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$
- The resulting parametric equations are

$$\begin{cases} x_1 = a_1 t + b_1 \\ x_2 = a_2 t + b_2 \\ \vdots \\ x_n = a_n t + b_n \end{cases} \quad t \in \mathbb{R}$$

Outline

Example

- Find the parametric equations of the line through $(1, -2, 3)$ and parallel to the vector $\pi\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

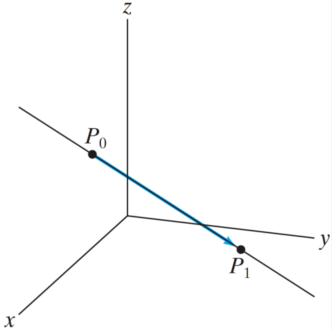
- $\mathbf{a} = \pi\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
- $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
- $\mathbf{r}(t) = \mathbf{i}-2\mathbf{j}+3\mathbf{k}+t(\pi\mathbf{i}-3\mathbf{j}+\mathbf{k}) = (1+\pi t)\mathbf{i}+(-2-3t)\mathbf{j}+(3+t)\mathbf{k}$
- The parametric equations may be read as

$$\begin{cases} x = \pi t + 1 \\ y = \; -3t - 2 \qquad t \in \mathbb{R} \\ z = t + 3 \end{cases}$$

Example

From Euclidean geometry, two distinct points determine a unique line in $\mathbb{R}^2$ or $\mathbb{R}^3$

Find the parametric equations of the line through the points $P_0(1, -2, 3)$ and $P_1(0, 5, -1)$



Example

- We need to find a vector **a** parallel to the desired line
- The vector with tail at P_0 and head at P_1 is such a vector

$$\overrightarrow{P_0P_1} = (0-1, 5-(-2), -1-3) = -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$$

- For **b**, t he position vector of a particular point on the line, we have the choice of taking either

$$\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \quad \text{or} \quad \mathbf{b} = 5\mathbf{j} - \mathbf{k}$$

- Hence, the parametric equations

$$\begin{cases} x = 1 - t \\ y = -2 + 7t \\ z = 3 - 4t \end{cases} \quad t \in \mathbb{R} \quad \text{or} \quad \begin{cases} x = -t \\ y = 5 + 7t \\ z = -1 - 4t \end{cases} \quad t \in \mathbb{R}$$

Parametric Equations of Lines Through Two Distinct Points

- Given two arbitrary points $P_0(a_1, a_2, a_3)$ and $P_1(b_1, b_2, b_3)$
- The line joining them has vector parametric equation

$$\mathbf{r}(t) \quad = \quad \overrightarrow{OP_0} + t\overrightarrow{P_0P_1}$$

- Which gives parametric equations

$$\begin{cases} x = a_1 + (b_1 - a_1)t \\ y = a_2 + (b_2 - a_2)t \qquad t \in \mathbb{R} \\ z = a_3 + (b_3 - a_3)t \end{cases}$$

Note

Parametric equations for a line
(or, more generally, for any curve) are never unique.

From Parametric Equations To Symmetric Form of a Line

- Assume that each $a_i, \ i = 1, 2, 3$ is nonzero.
- One can eliminate the parameter variable t in each equation

$$\begin{cases} x = a_1 t + b_1 \\ y = a_2 t + b_2 \\ z = a_3 t + b_3 \end{cases} \quad t \in \mathbb{R} \ \Rightarrow \ \begin{cases} t = \frac{x-b_1}{a_1} \\ t = \frac{y-b_2}{a_2} \\ t = \frac{z-b_3}{a_3} \end{cases} \quad t \in \mathbb{R}$$

- Thus, the **symmetric form** is

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3}$$

Example 4

- The first set of parametric equations give rise to the corresponding symmetric form

$$\begin{cases} x = 1 - t \\ y = -2 + 7t \\ z = 3 - 4t \end{cases} \quad t \in \mathbb{R} \;\Rightarrow\; \begin{cases} t = \frac{x-1}{-1} \\ t = \frac{y+2}{7} \\ t = \frac{z-3}{-4} \end{cases} \quad t \in \mathbb{R}$$

$$\frac{x-1}{-1} = \frac{y+2}{7} = \frac{z-3}{-4}$$

Example

Find where the line with parametric equations

$$\begin{cases} x = t + 5 \\ y = \; -2t - 4 \qquad t \in \mathbb{R} \\ z = 3t + 7 \end{cases}$$

intersects the plane $3x + 2y - 7z = 2$

- We must locate the point of intersection.
- One way is to find what value of the parameter t gives a point on the line that also lies in the plane

Example

Find where the line with parametric equations

$$\begin{cases} x = t + 5 \\ y = -2t - 4 \qquad t \in \mathbb{R} \\ z = 3t + 7 \end{cases}$$

intersects the plane $3x + 2y - 7z = 2$

- This is accomplished by substituting the parametric values for x, y, and z from the line into the equation for the plane.

$$3(t + 5) + 2(-2t - 4) - 7(3t + 7) = 2$$

- Solving the equation for t, we find that $t = -2$.

Example

Find where the line with parametric equations

$$\begin{cases} x = t + 5 \\ y = -2t - 4 \qquad t \in \mathbb{R} \\ z = 3t + 7 \end{cases}$$

intersects the plane $3x + 2y - 7z = 2$.

- Setting t equal to -2 in the parametric equations for the line yields the point $(3, 0, 1)$.
- Point $(3, 0, 1)$, indeed, lies in the plane as well.

How should we do this if we start with the symmetric form of the line ?

Example 6

Determine whether and where the two lines

$$\begin{cases} x = t + 1 \\ y = 5t + 6 \\ z = -2t \end{cases} \quad t \in \mathbb{R} \;\text{ and }\; \begin{cases} x = 3t - 3 \\ y = t \\ z = t + 1 \end{cases} \quad t \in \mathbb{R}$$

intersect

- We must be able to find t_1 and t_2 so that, by equating the respective parametric expressions for x, y and z we have

$$\begin{cases} t_1 + 1 = 3t_2 - 3 \\ 5t_1 + 6 = t_2 \\ -2t_1 = t_2 + 1 \end{cases} \quad t \in \mathbb{R}$$

Example 6

Determine whether and where the two lines

$$\begin{cases} x = t + 1 \\ y = 5t + 6 \\ z = -2t \end{cases} \quad t \in \mathbb{R} \quad \text{and} \quad \begin{cases} x = 3t - 3 \\ y = t \\ z = t + 1 \end{cases} \quad t \in \mathbb{R}$$

intersect

$$\begin{cases} t_1 + 1 = 3t_2 - 3 \\ 5t_1 + 6 = t_2 \\ -2t_1 = t_2 + 1 \end{cases} \quad t \in \mathbb{R}$$

- Using the last two equations

$$t_2 = 5t_1 + 6 = -2t_1 - 1 \Rightarrow t_1 = -1$$

Example 6

Determine whether and where the two lines

$$\begin{cases} x = t + 1 \\ y = 5t + 6 \\ z = -2t \end{cases} \quad t \in \mathbb{R} \quad \text{and} \quad \begin{cases} x = 3t - 3 \\ y = t \\ z = t + 1 \end{cases} \quad t \in \mathbb{R}$$

intersect

$$\begin{cases} t_1 + 1 = 3t_2 - 3 \\ 5t_1 + 6 = t_2 \\ -2t_1 = t_2 + 1 \end{cases} \quad t \in \mathbb{R}$$

- Using $t_1 = -1$ in the second equation, we find that $t_2 = 1$
- Note that the values $t_1 = -1$ and $t_2 = 1$ also satisfy the first equation

Example 6

Determine whether and where the two lines

$$\begin{cases} x = t + 1 \\ y = 5t + 6 \\ z = -2t \end{cases} \quad t \in \mathbb{R} \quad \text{and} \quad \begin{cases} x = 3t - 3 \\ y = t \\ z = t + 1 \end{cases} \quad t \in \mathbb{R}$$

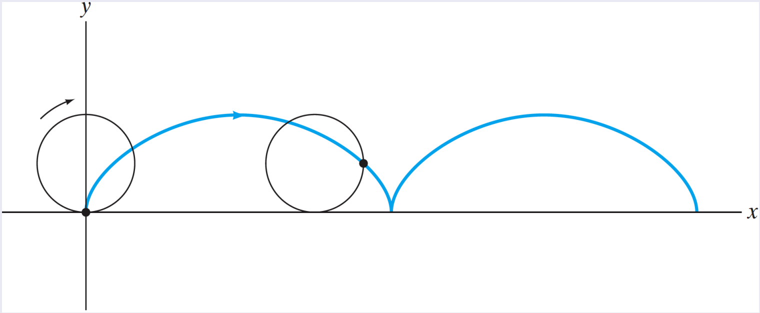
intersect

$$\begin{cases} t_1 + 1 = 3t_2 - 3 \\ 5t_1 + 6 = t_2 \\ -2t_1 = t_2 + 1 \end{cases} \quad t \in \mathbb{R}$$

- Setting $t = 1$ in the set of parametric equations for the first line gives the desired intersection point, namely, $(0, 1, 2)$.

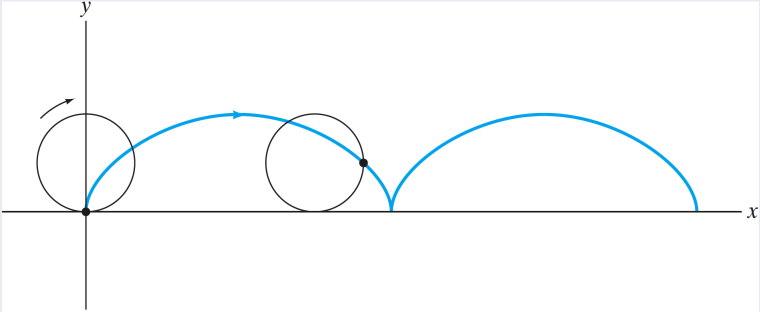
Example 7

- Assume a wheel rolls along a flat surface without slipping
- A point on the rim of the wheel traces a curve called a **cycloid**



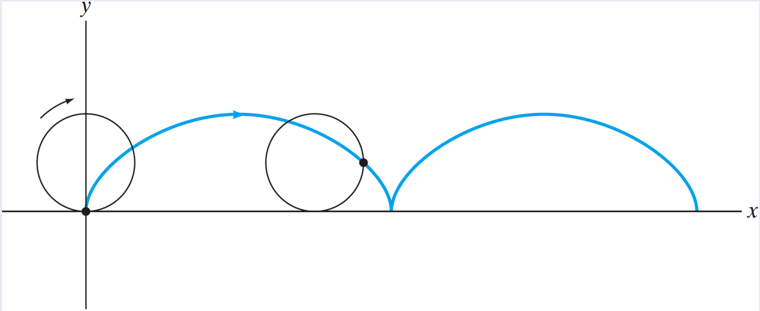
- Vector geometry makes it relatively easy to find parametric equations

Example 7



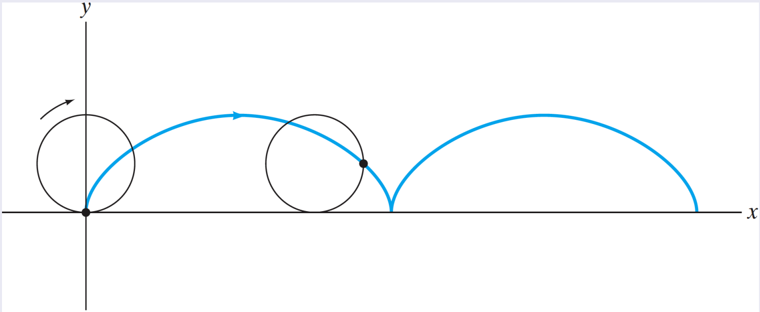
- Suppose that the wheel has radius a
- Suppose that coordinates in $\mathbb{R}^2$ are chosen so that the point of interest on the wheel is initially at the origin

Example 7



- After the wheel has rolled through a central angle of t radians, the situation is as shown in figure

Example 7



- The parametric equations are

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \quad t \in \mathbb{R}$$