Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 4

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Outline

1. New Coordinate Systems
   - Introduction
   - Polar Coordinates
   - Cylindrical coordinates
   - Spherical coordinates
   - Formula of conversion between coordinates systems and examples

Outline

Alternatives to Cartesian Coordinate Systems

- The Cartesian coordinate system will continue to be of **prime** importance to us.
- But from time to time, we will find it advantageous to use different coordinate systems.
- In $\mathbb{R}^2$, polar coordinates are useful for describing figures with circular symmetry.
- In $\mathbb{R}^3$, there are two other particularly valuable coordinate systems:
  - Cylindrical coordinates
  - Spherical coordinates
- As we shall see,

  Cylindrical and spherical coordinates, are each a way, of adapting polar coordinates for use in three dimensions

Motivation

As told rectangular or Cartesian coordinates are the most common and somehow natural way of representing point and figures. Nevertheless, sometimes is useful to use another system of coordinates. For example:

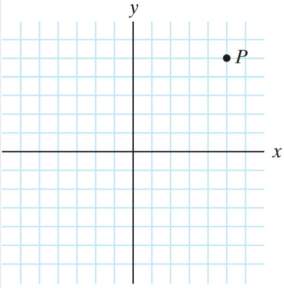
- **Finding limits**. We will see it in next chapter.
- **Solving integrals**. For example, computing the formula of the volume of a cone by means of integrals. We will do it on the correspondent chapter.
- **Simplifying** formulas and descriptions of systems. For example, the modelling of the movement of a mechanical arm. You have an example on, `https://tsc.urjc.es/~staffetti/robots.html`

Outline

Cartesian Coordinates on $\mathbb{R}^2$

We can understand the Cartesian (or rectangular) coordinates (x, y) of a point P in $\mathbb{R}^2$ in the following way:

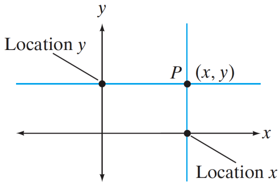
- Imagine the entire plane filled with horizontal and vertical lines:



- Then, the point P lies on exactly one vertical line and one horizontal line

Cartesian Coordinates on $\mathbb{R}^2$

- The x-coordinate of P is where this vertical line intersects the x-axis
- The y-coordinate is where the horizontal line intersects the y-axis

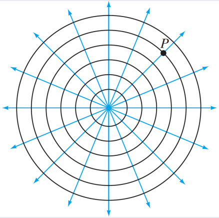


Because of this geometry, every point in $\mathbb{R}^2$
has a uniquely determined set of Cartesian coordinates

Polar Coordinates on $\mathbb{R}^2$

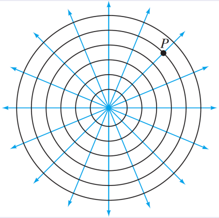
Polar coordinates are defined by considering different geometric information

- Imagine the plane filled with
  - Concentric circles centred at the origin, and
  - Rays emanating from the origin



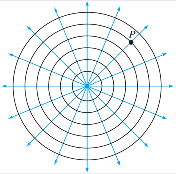
Every point in $\mathbb{R}^2$ except the origin lies on exactly one such circle and one such ray coordinates

Polar Coordinates on $\mathbb{R}^2$

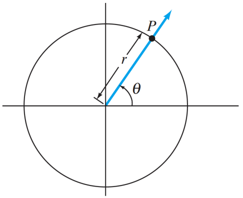


- The origin itself is special:
    - No circle passes through it.
    - All the rays begin at it.

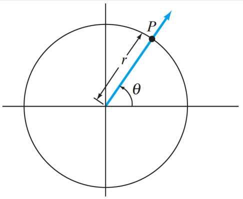
Polar Coordinates on $\mathbb{R}^2$



- For points P other than the origin, we assign to P the polar coordinates (r, θ)
  - r is the **radius** of the circle on which P lies
  - θ is the **angle** between the positive x-axis and the ray on which P lies.



Polar Coordinates on $\mathbb{R}^2$

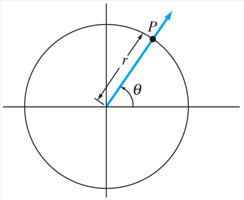


- θ is measured as opening counterclockwise.
- The origin is an exception:

  it is assigned the polar coordinates $(0, \theta)$,
  where θ can be any angle

- As we have described polar coordinates, $r \geq 0$ since r is the radius of a circle.

Polar Coordinates on $\mathbb{R}^2$



- It also makes good sense to require $0 \leq \theta < 2\pi$

  Then, every point in the plane, except the origin, has a uniquely determined pair of polar coordinates

- Occasionally, however, it is useful not to restrict r to be non-negative and θ to be between 0 and 2π (e.g. circular movement of a particle).

Polar Coordinates on $\mathbb{R}^2$

- Assume r and θ are not to restrict.

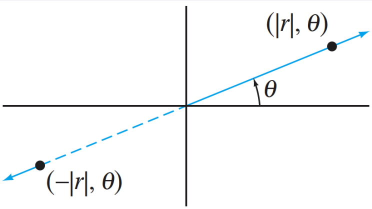
No point of $\mathbb{R}^2$ will be described by
a unique pair of polar coordinates

- If P has polar coordinates (r, θ), then it also has coordinates:
  - $(r, \theta + 2n\pi)$, and
  - $(-r, (\theta + \pi) + 2n\pi)$

where n can be any integer.

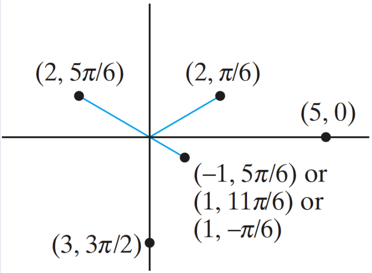
Polar Coordinates on $\mathbb{R}^2$

- To locate the point having coordinates (r, θ), where $r < 0$
  - Construct the ray making angle θ with respect to the **positive** x-axis, and
  - Instead of marching $|r|$ units away from the origin along this ray, go $|r|$ units in the opposite direction



Example 1

Make sure you understand that the points pictured in figure have the coordinates indicated:



Example 2

Graph the curve given by the polar equation

$$r = 6\cos\theta$$

- We can get a feeling for the graph by compiling values

θ	$r = 6\cos\theta$
0	6
$\pi/6$ | 30°	$3\sqrt{3}$
$\pi/4$ | 45°	$3\sqrt{2}$
$\pi/3$ | 60°	3
$\pi/2$ | 90°	0
$2\pi/3$ | 120°	-3
$3\pi/4$ | 135°	$-3\sqrt{2}$
$5\pi/6$ | 150°	$-3\sqrt{3}$
π	-6

θ	$r = 6\cos\theta$
$7\pi/6$ | 210°	$-3\sqrt{3}$
$5\pi/4$ | 225°	$-3\sqrt{2}$
$4\pi/3$ | 240°	-3
$3\pi/2$ | 270°	0
$5\pi/3$ | 300°	3
$7\pi/4$ | 315°	$3\sqrt{2}$

Example 2

Graph the curve given by the polar equation

$$r = 6\cos\theta$$

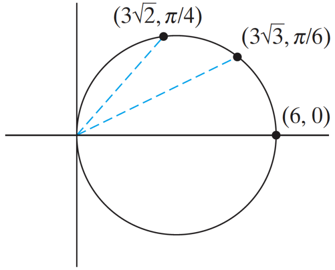
- r decreases from 6 to 0 as θ increases from 0 to $\pi/2$
- r decreases from 0 to -6 (or is not defined, if you take r to be nonnegative) as θ varies from $\pi/2$ to π
- r increases from -6 to 0 as θ varies from π to $3\pi/2$
- r increases from 0 to 6 as θ varies from $3\pi/2$ to 2π
- To graph the resulting curve, imagine a radar screen:

As θ moves counterclockwise from 0 to 2π , the point (r, θ) is traced as the appropriate "blip" on the radar screen

Example 2

Graph the curve given by the polar equation

$$r = 6\cos\theta$$



Example 2

Graph the curve given by the polar equation

$$r = 6\cos\theta$$

- Note that the curve is actually traced twice:
  - Once as θ varies from 0 to π, and
  - Then again as θ varies from π to 2π
- Alternatively, the curve is traced just once if we allow only θ values that yield nonnegative r values.

The resulting graph appears to be
a circle of radius 3 (not centred at the origin)

Conversions between polar and Cartesian coordinates

- Polar to Cartesian:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

- Cartesian to polar:

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$$

Example 3

Prove that the curve in Example 2

$$r = 6\cos\theta$$

really is a circle.

Example 3

- Multiply both sides of the equation by r

$$r^2 = 6r\cos\theta$$

- Conversions formulas immediately give

$$x^2 + y^2 = 6x$$

Example 3

Prove that the curve in Example 2

$$r = 6\cos\theta$$

really is a circle

Example 3

$$x^2 + y^2 = 6x$$

- We complete the square in x

$$(x-3)^2 + y^2 = 9$$

A circle of radius 3
with centre at $(3, 0)$

Outline

Generalizing polar coordinates to three dimensions

Cylindrical coordinates on $\mathbb{R}^3$ are a "naive" way of generalizing polar coordinates to three dimensions

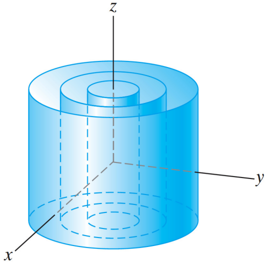
- They are nothing more than polar coordinates used in place of the x- and y-coordinates where

  The z-coordinate is left unchanged

- The geometry is as follows:

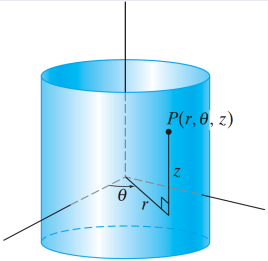
  "Fill all of space with infinitely extended circular cylinders with central axes the z-axis".

Generalizing polar coordinates to three dimensions



- Any point P in $\mathbb{R}^3$ not lying on the z-axis lies on exactly one such cylinder.

Generalizing polar coordinates to three dimensions



- The cylindrical coordinates of P are

$$(r, \theta, z)$$

It is just an extension of the **polar coordinates** to 3 dimension, $\mathbb{R}^3$.

Conversions between cylindrical and Cartesian coordinates

- Cylindrical to Cartesian:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

- Cartesian to cylindrical:

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$

- If we make restrictions then all points of $\mathbb{R}^3$ except the z-axis have a unique set of cylindrical coordinates

$$r \geq 0, 0 \leq \theta < 2\pi$$

Conversions between cylindrical and Cartesian coordinates

- Cylindrical to Cartesian:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

- Cartesian to cylindrical:

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$

- A point on the z-axis with Cartesian coordinates $(0, 0, z_0)$ has cylindrical coordinates

$$(0, \theta, z_0)$$

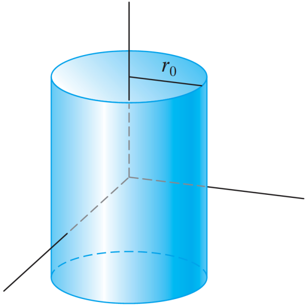
where θ can be any angle.

Symmetry and Cylindrical Coordinates

Cylindrical coordinates are useful for studying objects possessing an axis of symmetry

- Let's understand the three "constant coordinate" surfaces

1. The $r = r_0$ surface is just a cylinder of radius r_0 with axis the z-axis

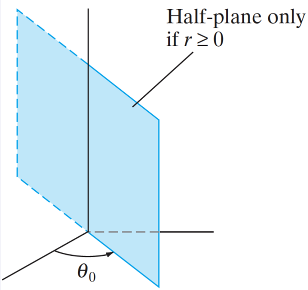


Symmetry and Cylindrical Coordinates

Cylindrical coordinates are useful for studying objects possessing an axis of symmetry

- Let's understand the three "constant coordinate" surfaces

2. The $\theta = \theta_0$ surface is a vertical plane containing the z-axis (or a half-plane with edge the z-axis if we take $r \geq 0$ only)

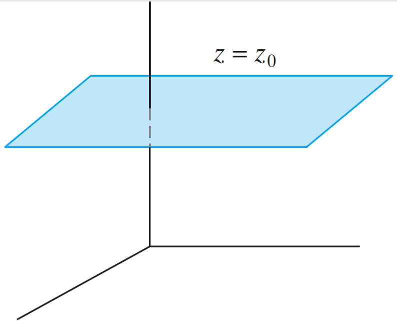


Symmetry and Cylindrical Coordinates

Cylindrical coordinates are useful for studying objects possessing an axis of symmetry

- Let's understand the three "constant coordinate" surfaces

3. The $z = z_0$ surface is a horizontal plane

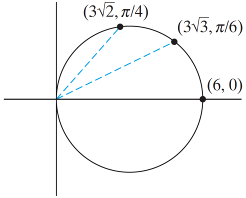


Example 4

Graph the surface having cylindrical equation

$$r = 6\cos\theta$$

- This equation is identical to the one in Example 2.
- In particular, z does not appear in this equation
- If the surface is sliced by the horizontal plane $z = c$, where c is a constant, we will see the circle



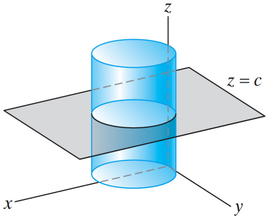
No matter what c is

Example 4

Graph the surface having cylindrical equation

$$r = 6\cos\theta$$

- If we stack these circular sections, then the entire surface is a circular cylinder



- This cylinder has radius 3 with axis parallel to the z-axis

Example 5

Graph the surface having equation in cylindrical coordinates

$$z = 2r$$

- The variable θ does not appear in the equation

The surface will be
circularly symmetric about the z-axis

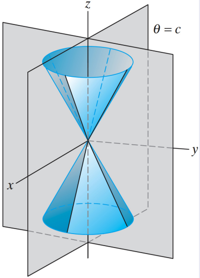
- If we slice the surface by any plane of the form $\theta =$ constant we see the same curve, namely, a **line of slope** 2.

Example 5

Graph the surface having equation in cylindrical coordinates

$$z = 2r$$

- As we let the constant-θ plane vary, this line generates a cone



- The cone consists only of the top half (nappe) when we restrict r to be nonnegative

Example 5

- Graph the surface having equation $z = 2r$ in cylindrical coordinates

- The Cartesian equation of this cone is readily determined using the conversion formulas

$$z = 2r \Rightarrow z^2 = 4r^2 \iff z^2 = 4(x^2 + y^2)$$

- Since z can be positive as well as negative, this last Cartesian equation describes the cone with both parts.
- If we want the top part only, then the equation is:

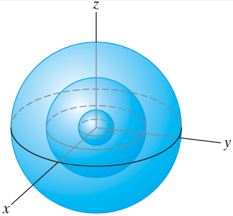
$$z = 2\sqrt{x^2 + y^2}$$

- Similarly, $z = -2\sqrt{x^2 + y^2}$ describes the bottom part.

Outline

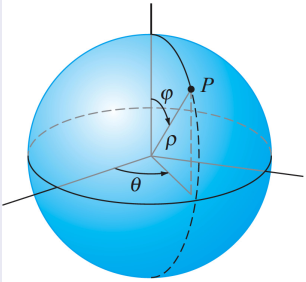
Space and spheres

- Fill all of space with spheres centered at the origin



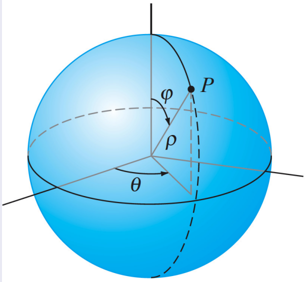
- Every point $P \in \mathbb{R}^3$, except the origin, lies on a single such sphere
- The spherical coordinates of P are given by specifying
  - The radius ρ of the sphere containing P, and
  - The latitude and longitude readings of P along this sphere

Space and spheres



- The spherical coordinates (ρ, φ, θ) of P are defined as:
  - ρ is the distance from P to the origin.
  - φ is the angle between the positive z-axis and the ray through the origin and P.
  - θ is the angle between the positive x-axis and the ray made by dropping a perpendicular from P to the xy-plane.

Space and spheres



- The θ-coordinate is exactly the same as the θ-coordinate used in cylindrical coordinates

Warning: Physicists usually prefer to reverse the roles of φ and θ, as do some graphical software packages

Space and spheres

- It is standard practice to impose the following restrictions on the range of values for the individual coordinates

$$\rho \geq 0, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta < 2\pi$$

- Then, all points of $\mathbb{R}^3$, except those on the z-axis, have a uniquely determined set of spherical coordinates.
- **Points along the z-axis**, except for the origin, have coordinates of the form

$$(\rho_0, 0, \theta) \quad \text{or} \quad (\rho_0, \pi, \theta)$$

where ρ_0 is a positive constant and θ is arbitrary. The origin has spherical coordinates,

$$(0, \varphi, \theta) \qquad \text{where both } \varphi \text{ and } \theta \text{ are arbitrary.}$$

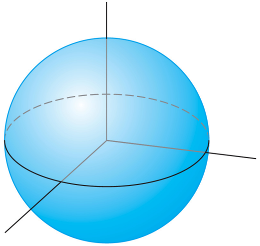
Spherical Coordinates and Symmetry

Spherical coordinates are especially useful for describing objects that have a centre of symmetry

- Let assume restrictions

$$\rho \geq 0, \;\; 0 \leq \varphi \leq \pi, \;\; 0 \leq \theta < 2\pi$$

1. The constant coordinate surface $\rho = \rho_0 \;\; (\rho_0 > 0)$ is a sphere of radius ρ_0



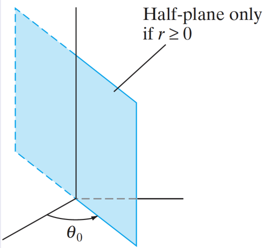
Spherical Coordinates and Symmetry

Spherical coordinates are especially useful for describing objects that have a center of symmetry

- Let assume restrictions

$$\rho \geq 0, \;\; 0 \leq \varphi \leq \pi, \;\; 0 \leq \theta < 2\pi$$

2. The surface given by $\theta = \theta_0$ is a half-plane just as in the cylindrical case



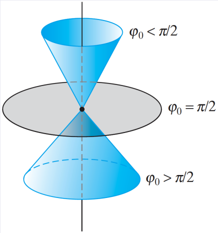
Spherical Coordinates and Symmetry

Spherical coordinates are especially useful for describing objects that have a center of symmetry

- Let assume restrictions

$$\rho \geq 0, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta < 2\pi$$

3. The $\varphi = \varphi_0$ surface is
   - A single-nappe cone if $\varphi_0 \neq \pi/2$
   - The xy-plane if $\varphi_0 = \pi/2$
   - The positive or negative z-axis if $\varphi_0 = 0$ or π, respectively



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1. New Coordinate Systems
   - Introduction
   - Polar Coordinates
   - Cylindrical coordinates
   - Spherical coordinates
   - Formula of conversion between coordinates systems and examples

Conversions Between Spherical and Cartesian Coordinates

- Spherical to Cartesian:

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

- Cartesian to Spherical:

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\varphi = \sqrt{x^2 + y^2}/z \\ \tan\theta = y/x \end{cases}$$

Conversions Between Spherical and Cylindrical Coordinates

- Spherical to cylindrical:

$$\begin{cases} r = \rho \sin \varphi \\ \theta = \theta \\ z = \rho \cos \varphi \end{cases}$$

- Cylindrical to Spherical:

$$\begin{cases} \rho^2 = r^2 + z^2 \\ \tan \varphi = r/z \\ \theta = \theta \end{cases}$$

Example 7

Convert to spherical equation the cylindrical equation in Example 5

$$z = 2r$$

- Using the conversion equations

$$\rho \cos \varphi = 2\rho \sin \varphi$$

- Therefore

$$\tan \varphi = \frac{1}{2} \iff \varphi = \tan^{-1} \frac{1}{2} \approx 26^\circ$$

Thus, the equation defines a cone

- The spherical equation is especially simple since it involves just a single coordinate

Example 8

Not all spherical equations are improvements over their cylindrical or Cartesian counterparts

- Consider the Cartesian equation

$$6x = x^2 + y^2$$

- The polar-cylindrical equivalent equation is

$$6r\cos\theta = r^2 \rightarrow r = 6\cos\theta$$

- Using the conversion equations

$$6\rho\sin\varphi\cos\theta = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta$$

Example 8

- Using the conversion equations

$$6\rho \sin\varphi \cos\theta = \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta$$

- Simplifying

$$6\rho \sin\varphi \cos\theta = \rho^2 \sin^2\varphi(\cos^2\theta + \sin^2\theta)$$
$$\iff 6\rho \sin\varphi \cos\theta = \rho^2 \sin^2\varphi \iff 6\cos\theta = \rho \sin\varphi$$

- This spherical equation is more complicated than the original Cartesian equation

All three spherical coordinates
are involved

- It is not obvious that the spherical equation describes a cylinder.

Example 9

Graph the surface with spherical equation

$$\rho = 2a\cos\varphi \quad \text{where } a > 0$$

- Note that the equation is independent of θ.
- The same hold for the graph of the cone with cylindrical equation

$$z = 2r$$

- Thus, all sections of this surface made by slicing with the half-plane $\theta = c$ must be the same.

Example 9

Graph the surface with spherical equation

$$\rho = 2a\cos\varphi \quad \text{where } a > 0$$

- We compile values as in the adjacent table

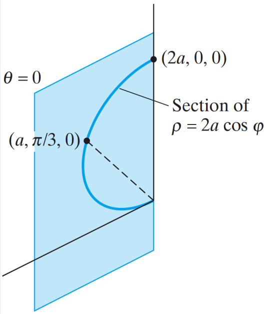
φ	$\rho = 2a\cos\varphi$
0	$2a$
$\pi/6$	$\sqrt{3}a$
$\pi/4$	$\sqrt{2}a$
$\pi/3$	a
$\pi/2$	0
$2\pi/3$	$-a$
$3\pi/4$	$-\sqrt{2}a$
π	$-2a$

Example 9

Graph the surface with spherical equation

$$\rho = 2a\cos\varphi \quad \text{where } a > 0$$

- Then, the section of the surface in the half-plane $\theta = 0$ is as shown in figure

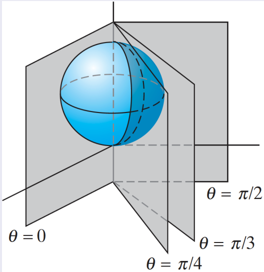


Example 9

Graph the surface with spherical equation

$$\rho = 2a\cos\varphi \quad \text{where } a > 0$$

- This section must be identical in all other constant-θ half-planes
- Then, this surface appears to be a sphere of radius a tangent to the xy-plane



Example 9

Determine the Cartesian equation corresponding to the spherical equation
$$\rho = 2a\cos\varphi \quad \text{where } a > 0$$

- Multiplying both sides of the spherical equation by ρ and using the conversion equations

$$\rho = 2a\cos\varphi \Rightarrow \rho^2 = 2a\rho\cos\varphi \iff x^2 + y^2 + z^2 = 2az$$
$$\iff x^2 + y^2 + (z-a)^2 = a^2$$

The equation of a sphere of radius a
with center at $(0, 0, a)$ in Cartesian coordinates