Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica
Lecture 8

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Outline

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1. Properties; Higher-order Partial Derivatives
   - Properties of Differentiation
   - kth order derivatives and Schwarz Theorem

Differentiation is a linear operation:

Proposition 4.1: Linearity of Differentiation

- Let $\mathbf{f}, \mathbf{g} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be two functions that are both differentiable at a point $\mathbf{a} \in X$ and let $c \in \mathbb{R}$ be any scalar.
- Then,
  1. The function $\mathbf{h} = \mathbf{f} + \mathbf{g}$ is also differentiable at $\mathbf{a}$, and
  $$D\mathbf{h}(\mathbf{a}) = D(\mathbf{f} + \mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a})$$
  2. The function $\mathbf{k} = c\mathbf{f}$ is differentiable at $\mathbf{a}$, and
  $$D\mathbf{k}(\mathbf{a}) = D(c\mathbf{f})(a) = cD\mathbf{f}(\mathbf{a})$$

Example 1

- Let $\mathbf{f}$ and $\mathbf{g}$ be defined by,

$$\begin{aligned} \mathbf{f}(x,y) &= (x+y, xy\sin y, y/x) \\ \mathbf{g}(x,y) &= (x^2+y^2, ye^{xy}, 2x^3-7y^5) \end{aligned}$$

- Then

$$\begin{aligned} D\mathbf{f}(x,y) &= \begin{bmatrix} 1 & 1 \\ y\sin y & x\sin y + xy\cos y \\ -y/x^2 & 1/x \end{bmatrix} \\ D\mathbf{g}(x,y) &= \begin{bmatrix} 2x & 2y \\ y^2e^{xy} & e^{xy}+xye^{xy} \\ 6x^2 & -35y^4 \end{bmatrix} \end{aligned}$$

- $\mathbf{f}$ is differentiable only in $\mathbb{R}^2 \setminus \{x=0\}$ and $\mathbf{g}$ is differentiable on all of $\mathbb{R}^2$.

Example 1

- Let **f** and **g** be defined by

$$\begin{aligned} \mathbf{f}(x,y) &= (x+y, xy\sin y, y/x) \\ \mathbf{g}(x,y) &= (x^2+y^2, ye^{xy}, 2x^3-7y^5) \end{aligned}$$

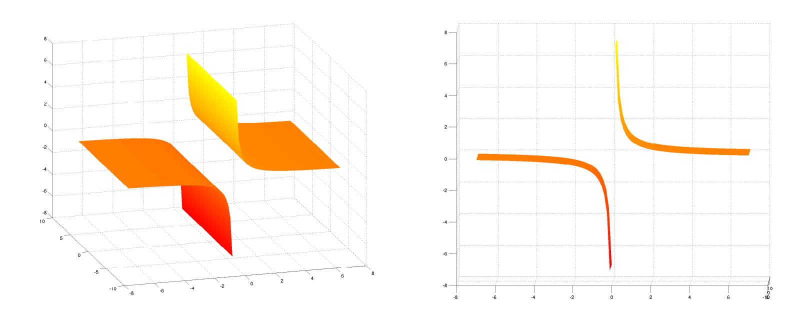
- If we let $\mathbf{h} = \mathbf{f} + \mathbf{g}$, then Proposition 4.1 tells us that **h** must be differentiable on all of its domain
- Furthermore,

$$\begin{aligned} D\mathbf{h}(x,y) &= D\mathbf{f}(x,y) + D\mathbf{g}(x,y) \\ &= \begin{bmatrix} 2x+1 & 2y+1 \\ y\sin y + y^2e^{xy} & x\sin y + xy\cos y + e^{xy} + xye^{xy} \\ 6x^2 - y/x^2 & 1/x - 35y^4 \end{bmatrix} \end{aligned}$$

Example 1

- Some graphical representation.

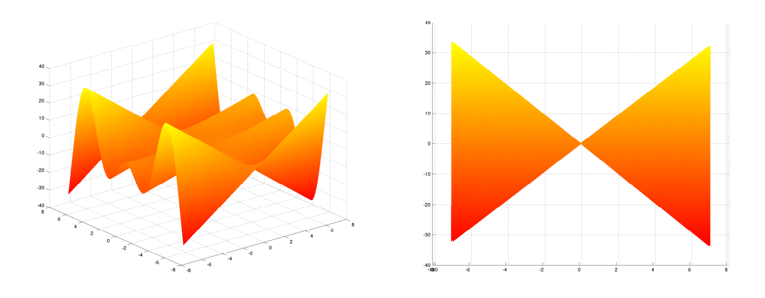
$$\frac{\partial f_3}{\partial y} = 1/x$$



Example 1

- Some graphical representation.

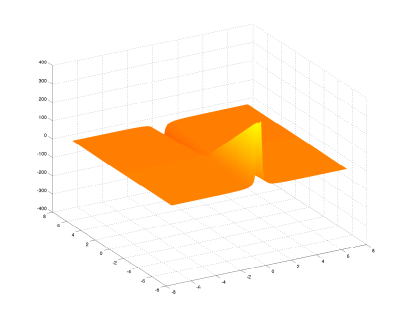
$$f_2 = xy \sin y$$

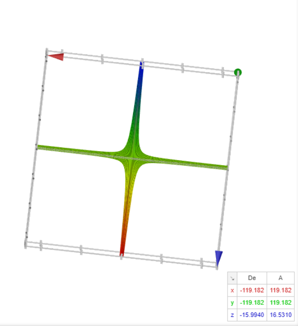


Example 1

- Some graphical representation.

$$\frac{\partial f_3}{\partial x} = -y/x^2$$





Proposition 4.2

- Let $f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in X$.
- Then,

1. The product function fg is also differentiable at $\mathbf{a}$:

$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a})$$

2. If $g(\mathbf{a}) \neq 0$, then the quotient function f/g is differentiable at $\mathbf{a}$:

$$D(f/g)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$$

Example 2

- Suppose

$$
\begin{aligned}
f(x,y,z) &= ze^{xy} \\
g(x,y,z) &= xy + 2yz - xz
\end{aligned}
$$

- Then

$$(fg)(x,y,z) = (xyz + 2yz^2 - xz^2)e^{xy}$$

Example 2

- Suppose

$$
\begin{aligned}
f(x,y,z) &= ze^{xy} \\
g(x,y,z) &= xy + 2yz - xz
\end{aligned}
$$

- Then

$$(fg)(x,y,z) = (xyz + 2yz^2 - xz^2)e^{xy}$$

- So that

$$D(fg)(x,y,z) = \begin{bmatrix} (yz - z^2)e^{xy} + (xyz + 2yz^2 - xz^2)ye^{xy} \\ (xz + 2z^2)e^{xy} + (xyz + 2yz^2 - xz^2)xe^{xy} \\ (xy + 4yz - 2xz)e^{xy} \end{bmatrix}^T$$

Example 2

$$\begin{aligned} f(x,y,z) &= ze^{xy} \\ g(x,y,z) &= xy + 2yz - xz \\ Df(x,y,z) &= \begin{bmatrix} yze^{xy} & xze^{xy} & e^{xy} \end{bmatrix} \\ Dg(x,y,z) &= \begin{bmatrix} y - z & x + 2z & 2y - x \end{bmatrix} \end{aligned}$$

- Using Proposition 4.2

$$g(x,y,z)Df(x,y,z) + f(x,y,z)Dg(x,y,z) =$$

$$= \begin{bmatrix} (xy^2z + 2y^2z^2 - xyz^2)e^{xy} \\ (x^2yz + 2xyz^2 - x^2z^2)e^{xy} \\ (xy + 2yz - xz)e^{xy} \end{bmatrix}^T + \begin{bmatrix} (yz - z^2)e^{xy} \\ (xz + 2z^2)e^{xy} \\ (2yz - xz)e^{xy} \end{bmatrix}^T$$

$$= e^{xy} \begin{bmatrix} (yz - z^2) + (xyz + 2yz^2 - xz^2)y \\ (xz + 2z^2) + (xyz + 2yz^2 - xz^2)x \\ (xy + 4yz - 2xz) \end{bmatrix}^T$$

Outline

How many **“second derivatives”** does a function have?

Example 3

- Let

$$f(x, y, z) = x^2y + y2z$$

- The first-order partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \\ \frac{\partial f}{\partial y} &= x^2 + 2yz \\ \frac{\partial f}{\partial z} &= y^2 \end{aligned}$$

Example 3

$$f(x,y,z) = x^2y + y2z$$
$$\frac{\partial f}{\partial x} = 2xy, \;\; \frac{\partial f}{\partial y} = x^2 + 2yz, \;\; \frac{\partial f}{\partial z} = y^2$$

- The second-order partial derivative with respect to x is,

$$f_{xx}(x,y,z) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy) = 2y$$

- Similarly, the second-order partial derivatives with respect to y and z are, respectively,

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x^2 + 2yz) = 2z$$
$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}(y^2) \equiv 0$$

Example 3

$$f(x,y,z) = x^2y + y2z$$
$$\frac{\partial f}{\partial x} = 2xy, \;\; \frac{\partial f}{\partial y} = x^2 + 2yz, \;\; \frac{\partial f}{\partial z} = y^2$$

- The mixed partial derivative with respect to first x and then y

$$f_{xy}(x,y,z) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy) = 2x$$

- There are **five more mixed** partials for this particular function

$$\frac{\partial^2 f}{\partial x \partial y}, \;\; \frac{\partial^2 f}{\partial z \partial x}, \;\; \frac{\partial^2 f}{\partial x \partial z}, \;\; \frac{\partial^2 f}{\partial z \partial y}, \;\; \frac{\partial^2 f}{\partial y \partial z}$$

General kth-order partial derivatives

- Suppose $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is a scalar-valued function of n variables.
- The kth-order partial derivative with respect to the variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ (in that order) is the iterated derivative

$$\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_2} \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial}{\partial x_{i_2}} \frac{\partial}{\partial x_{i_1}} (f(x_1, x_2, \ldots, x_n))$$

  where $i_1, i_2, \ldots, i_k$ are integers in the set $\{1, 2, \ldots, n\}$ (possibly repeated)
- Equivalent notation,

$$f_{x_{i_1} x_{i_2} \cdots x_{i_k}}(x_1, x_2, \ldots, x_n)$$

Example 4

- Let

$$f(x,y,z,w) = xyz + xy^2 w - \cos(x + zw)$$

- We then have

$$\begin{aligned}
f_{yw}(x,y,z,w) &= \frac{\partial^2 f}{\partial w \partial y} = \frac{\partial}{\partial w}\frac{\partial}{\partial y}(xyz + xy^2 w - \cos(x + zw)) \\
&= \frac{\partial}{\partial w}(xz + 2xyw) = 2xy \\
f_{wy}(x,y,z,w) &= \frac{\partial^2 f}{\partial w \partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial w}(xyz + xy^2 w - \cos(x + zw)) \\
&= \frac{\partial}{\partial y}(xy^2 + z\sin(x + zw)) = 2xy
\end{aligned}$$

This example suggests that there might be
a simple relationship among the mixed second partials

Theorem 4.3 (Schwarz)

- Suppose that X is open in $\mathbb{R}^n$.
- Suppose $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ has continuous first- and second-order partial derivatives.
- Then **the order** in which we evaluate the mixed second-order partials **is immaterial**.
- That is, if i_1 and i_2 are any two integers between 1 and n, then,

$$\frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}} = \frac{\partial^2 f}{\partial x_{i_2} \partial x_{i_1}}$$

Definition 4.4: Smooth Functions

- Assume X is open in $\mathbb{R}^n$.
- Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function.
- Function f is said to be of class C^k if its partial derivatives up to order at least k, exist and are continuous on X.
- Function f is said to be of class C^∞, or smooth, if it has continuous partial derivatives of all orders on X

A vector-valued function $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is of class $C^k(C^\infty)$

if and only if

Each of its component functions is of class $C^k(C^\infty)$

Theorem 4.5 Schwarz (extended)

- Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of class C^k
- Then the order in which we calculate any kth-order partial derivative does not matter
- Suppose
  - $(i_1, \ldots, i_k)$ are any k integers (not necessarily distinct) between 1 and n, and
  - $(j_1, \ldots, j_k)$ is any permutation (rearrangement) of these integers
- Then

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} = \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}$$

Example 5

- Let
$$f(x, y, z, w) = x^2 w e^{yz} - z e^{xw} + xyzw$$

- We verify Theorem 4.5
$$\frac{\partial^5 f}{\partial x \partial w \partial z \partial y \partial x} = 2e^{yz}(yz + 1) = \frac{\partial^5 f}{\partial z \partial y \partial w \partial^2 x}$$