Chapter 7: Analysis of functions and their graphical representation

Michael Stich

November 26, 2020

7 Analysis of functions and their graphical representation

In this chapter we apply the methods we have studied in the previous chapters to analyse functions and in particular to perform graphical representations.

7.1 Asymptotes, monotonicity, and curvature of a function

7.1.1 Monotonicity and curvature

Theorem 1 (graphical characterization of differentiable functions I):
Let f be a function continuous in $[a, b]$ and differentiable in (a, b).
(i) If $f'(x) \geq 0$ for $x \in (a, b)$, then f is (monotonically) increasing in (a, b).
(ii) If $f'(x) \leq 0$ for $x \in (a, b)$, then f is (monotonically) decreasing in (a, b).
(iii) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant in (a, b).

Comment:
If in (i) the $\geq$ sign is replaced by $>$, and in (ii) the $\leq$ sign is replaced by $<$, then we refer to the function as *strictly* monotonically in- or decreasing. Often, and in particular when it is clear from the context, the word *monotonically* is then suppressed.

Theorem 2 (second derivative):
Let f be differentiable with $f'(x) = g(x)$. If the derivative of $g(x)$ exists (according to

position in an interval of time. The second derivative is therefore the "rate of change" of the (first) derivative. For example, the change of velocity in an interval of time represents the acceleration of the body in motion.

In this way, higher derivatives can be introduced:

Definition 1 (higher derivatives):
Let f be n-times differentiable. Then, $f^{(n)}(x) = \dfrac{d^n f}{dx^n}(x) = \dfrac{d^n f(x)}{dx^n}$ represents the n-th derivative of f.

Higher derivatives can be applied, e.g. using L'Hôpital's rule repeatedly when an indetermination is replicated, or in the analysis of curves. For the latter, we need additional definitions:

Definition 2 (critical point):
(i) Let f be a non-constant function, differentiable in c with $f'(c) = 0$. Then, c is a critical point of f.
(ii) Let f be a function with domain D. If f is not differentiable in $x = c \in D$, c is a critical point of f.

Example 1: Find the critical points of $f(x) = x^3 - 9x^2 + 24x - 10$.

f is a polynomial and as such differentiable in $\mathbb{R}$. Its critical points are the solutions of $f'(c) = 0$: $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4) = 0$, whose solutions are $c = 2$ and $c = 4$.

Example 2: Find the critical points of $f(x) = |x|$.

We have seen this function earlier. Specifically, $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. There is no critical point according to Def. 2(i), but one according to Def. 2(ii) since f is not differentiable in $x = 0$. There is a critical point in $x = 0$.

Theorem 3 (extrema: criterion of the first derivative):
Let f be a differentiable function and c a critical point of f. Then:
(i) If $f'(x)$ changes from negative to positive in c, there is a local minimum in $(c, f(c))$.
(ii) If $f'(x)$ changes from positive to negative in c, there is a local maximum in $(c, f(c))$.

x only. Nevertheless, we know that in a graph, an extremum (or any other point on the curve) is characterized by a set of two numbers, c and $f(c)$. There are contexts where it is important to state both numbers.

Example 3: $f(x) = |x|$.

We know from Ex. 2 that f has a critical point in $x = 0$ and that f is differentiable in $\mathbb{R} \setminus 0$. We know that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. Following Theorem 3, $f(x) = |x|$ has a local minimum in $x = 0$.

Example 4: $f(x) = x^3$.

The function f is non-constant and differentiable in $\mathbb{R}$. The derivative of f is $f'(x) = 3x^2$. Then, the condition $f'(c) = 0$ is verified for $c = 0$, critical point of f. Nevertheless, $f'(x) = 3x^2 > 0$ for $x < 0$ and also for $x > 0$, i.e., $f'(x)$ does not change sign in the critical point and there is no local extremum according to Theorem 3 (but compare with the comment on the theorem of extrema in the last chapter).

In the following theorem, we introduce fundamental notions about the curvature of a function (convex/concave) and characterize *inflexion points* that separate areas of convex and concave curvatures. These concepts rely on the signs of the second derivative, in a similar way as the signs of the first derivative indicate the growth of a function (increase/decrease) and establish extrema as points that separate increasing from decreasing functions.

Theorem 4 (graphical caracterization of differentiable functions II):
Let $f(x)$ be a function whose second derivative $f''(x)$ exists in (a, b).
(i) If $f''(x) > 0$ for $x \in (a, b)$, then f is convex in (a, b).
(ii) If $f''(x) < 0$ for $x \in (a, b)$, then f is concave in (a, b).
(iii) If $f''(c) = 0$ for $c \in (a, b)$ and $f''(x)$ changes its sign in $x = c$, then f has an inflexion point in c.

Example 5: We consider the function of Ex. 4: $f(x) = x^3$. Its second derivative is $f''(x) = 6x$. According to Theorem 4(iii), in $c = 0$ the second derivative is zero, and furthermore $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, i.e., f'' changes its sign and in $x = 0$ we find an inflexion point. According to (i) and (ii), the function changes

$f'(x)$ is strictly decreasing.

(3) In a slight abuse of notation, the curvature of a graph is also referred to as its "convexity" (although the shape may be concave).

(4) Another example: If f represents the daytime (duration of sun visible above the horizon), then f' represents the change of daytime. After the summer solstice, nights are getting longer ($f'(x) < 0$) until the winter solstice (i.e., from June to December in the Northern hemisphere). Nevertheless, the *rate* with which the daylight changes grows only from the summer solstice to the autumn equinox ($f''(x) > 0$), and later decreases from the autumn equinox to the winter solstice ($f''(x) < 0$). The solstices represent the local extrema and the equinoxes the inflexion points.

Theorem 5 (extrema: criterion of the second derivative):
Let c be a critical point of f. If $f''(c)$ exists, then:
(i) If $f''(c) > 0$, then there is a local minimum in $x = c$;
(ii) If $f''(c) < 0$, then there is a local maximum in $x = c$;
(iii) If $f''(c) = 0$, the criterion is not conclusive (there may be a minimum, maximum or neither).

Example 6: Let $f(x) = x^2$. We calculate the first derivative and require it to vanish: $f'(c) = 0$, implying that $2c = 0$. Therefore, there is a critical point in $x = 0$. We can use Theorem 3 to establish that there is a local minimum in $x = 0$, but we can also use Theorem 5: $f''(x) = (2x)' = 2$. For all x, $2 > 0$, and specifically $f''(0) > 0$ and there is a local minimum in $x = 0$.

Example 7: Let us consider the function from examples 4 and 5: $f(x) = x^3$. The second derivative is $f''(x) = 6x$. According to Theorem 5(iii), in $x = 0$ (locus of the critical point) the second derivative vanishes and the criterion is not conclusive. Actually, we already established in Ex. 5 that in $x = 0$ there is an inflexion point and there is no contradiction.

7.1.2 Asymptotes

Asymptotic behavior of functions was already considered in the chapter on limits. In particular, when $\lim_{x\to c} = \infty$ or $\lim_{x\to c} = -\infty$ (or for lateral limits, for that matter), the curve of the function shows a *vertical asymptote*.

limit value $L = 0$. Consequently, the curve of e^{-x} approaches the *horizontal asymptote* $L = 0$ as $x \to \infty$. Of course, horizontal asymptotes can also be found for $x \to -\infty$, depending on the function.

7.2 Graphical representation of a function

We have now at our disposal the most fundamental tools for sketching functions graphically:

1. Domain. We know that the domain is typically an interval being a subset of $\mathbb{R}$, but a function domain can also be the union of various intervals, and can exclude points.

2. Symmetries. Functions may show a fundamental symmetry like $f(x) = f(-x)$ (even function, e.g., x^2, $\cos(x)$), $f(x) = -f(-x)$ (odd function, e.g., x^3, $\sin(x)$), or $f(x) = f(x + c)$, $c \neq 0$ (periodic function, e.g. $\sin(x)$ with $c = 2\pi$).

3. Discontinuities. We know how to detect removable, jump, and essential (including infinite) discontinuities and how they are represented.

4. Points of non-differentiability (for continuous functions). For example, $f(x) = |x|$ is not differentiable at $x = 0$, with the graph showing a peak there. Points of non-differentiability often show peaks or *cusps* if the lateral limits of the differential quotient do not coincide.

5. Intersection points with axes: The condition $x = 0$ gives the intersection with the ordinate, the condition $f(x) = 0$ the intersection(s) with the abscissa, also called roots or zeros.

6. Growth and local extrema. This is evaluated with help of the first derivative (increasing for $f'(x) > 0$, decreasing for $f'(x) < 0$, and an extremum for $f'(x)$ changing sign). The critical points are the points of non-differentiability and those which fulfill $f'(x) = 0$.

7. Curvature and inflexion points. This is evaluated with the help of the second derivative (convex for $f''(x) > 0$, concave for $f''(x) < 0$, and an inflexion point for $f''(x)$ changing sign).

8. Asymptotes. Horizontal asymptotes with value L correspond to solutions of $\lim f(x) = L$ (only if the domain of the function extends to $\pm\infty$) and vertical