Chapter 4 - MATRIX ALGEBRA

4.1. Matrix Operations

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{i1} & a_{i2} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\ \vdots & \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

• The entry in the ith row and the jth column of a matrix A is refered to as $(A)_{ij}$.

EXAMPLE:

• A **zero matrix** is a matrix, written 0, whose entries are all zero.

• A **square** matrix has the same number of rows than columns.

• In general $(m \neq n)$, matrices are **rectangular**.

• The **(main) diagonal** of a matrix, or its **diagonal entries**, are the entries

• A **diagonal matrix** has all its nondiagonal entries equal to zero.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• A matrix is **upper triangular** if all its elements under the diagonal are zero

• A matrix is **lower triangular** if all its elements over the diagonal are zero

• The set of all possible matrices of dimension $(m \times n)$ whose entries are real numbers is refered to as $\mathbb{R}^{m\times n}$

• The set of all possible matrices of dimension $(m \times n)$ whose entries are complex numbers is refered to as $\mathbb{C}^{m\times n}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 2 \\ 7 & 1 \\ 3 & -3 \end{bmatrix} \in \mathbb{K}^{3\times 2}$$

• **OPERATIONS:**

Only for matrices with the <u>same dimensions</u>:

◦ **Equality**. Two matrices are equal if and only if their corresponding entries are equal.

$$\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} & \\ & \end{bmatrix} \neq \begin{bmatrix} & \\ & \end{bmatrix}$$

◦ **Addition**. A matrix whose entries are the sum of the corresponding entries of the matrices.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

◦ **Scalar Multiplication**. A matrix whose entries are the corresponding entries of the matrix multiplied by the scalar.

$$2\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

• **PROPERTIES:**

Let A, B and C be matrices of $\mathbb{K}^{m\times n}$ and λ, $\mu \in \mathbb{K}$:

◦ $A+B=B+A$	◦ $\lambda\,(A+B)=\lambda\,A+\lambda\,B$
◦ $A+(B+C)=(A+B)+C$	◦ $(\lambda+\mu)\,A=\lambda\,A+\mu\,A$
◦ $A+0=A$	◦ $\lambda\,(\mu\,A)=(\lambda\,\mu)\,A$

Matrix Multiplication

$\boxed{\mathbb{K}^p}$ $\boxed{\mathbb{K}^n}$ $\boxed{\mathbb{K}^m}$

One wonders:

Does C exist $\mid$ $C\,\mathbf{x} = A\,B\,\mathbf{x} \quad \forall\,\mathbf{x}\in\mathbb{K}^p$?

PROBLEM: *What dimensions would C have?*

If we write $B=[\,\mathbf{b}_1\;\mathbf{b}_2\;\ldots\;\mathbf{b}_p\,]$ and $\mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$, then:

$$B\mathbf{x} \;=\; x_1\,\mathbf{b}_1 + x_2\,\mathbf{b}_2 + \cdots + x_p\,\mathbf{b}_p$$

$$A(B\mathbf{x}) \;=$$

$$=$$

$$=$$

$$= \qquad\qquad =$$

• Let A be an $(m\times n)$ matrix and let B be an $(n\times p)$ matrix with columns $\mathbf{b}_1$, $\mathbf{b}_2$, …, $\mathbf{b}_p$. The **matrix product** of A by B is the $(m\times p)$ matrix AB whose columns are $A\mathbf{b}_1$, $A\mathbf{b}_2$, …, $A\mathbf{b}_p$.

That is,

$$AB = A\,[\,\mathbf{b}_1\;\mathbf{b}_2\;\ldots\;\mathbf{b}_p\,] = [\,A\mathbf{b}_1\;A\mathbf{b}_2\;\ldots\;A\mathbf{b}_p\,]$$

Warning: The dimensions of the matrices involved in a product must verify

A	B	$=$	C
( )	( )		( )

EXAMPLE:

$$\begin{bmatrix}2&3\\1&-5\end{bmatrix}\ \begin{bmatrix}4&3&6\\1&-2&3\end{bmatrix}\ =$$

$$(\qquad)\qquad(\qquad)\qquad\Rightarrow\ (\qquad)$$

$$=\left[\ \begin{bmatrix}2&3\\1&-5\end{bmatrix}\begin{bmatrix}\ \\ \ \end{bmatrix}\ \begin{bmatrix}2&3\\1&-5\end{bmatrix}\begin{bmatrix}\quad \\ \ \end{bmatrix}\ \begin{bmatrix}2&3\\1&-5\end{bmatrix}\begin{bmatrix}\ \\ \ \end{bmatrix}\ \right]=$$

$$=\left[\ \begin{bmatrix}\quad \\ \ \end{bmatrix}\ \begin{bmatrix}\quad \\ \ \end{bmatrix}\ \begin{bmatrix}\quad \\ \ \end{bmatrix}\ \right]=\begin{bmatrix}\qquad\qquad\qquad \\ \ \end{bmatrix}$$

Row-Column Rule for computing AB

Consider $A \in \mathbb{K}^{m\times n}$, and $B = [\ \mathbf{b}_1\ \dots\ \mathbf{b}_p\ ] \in \mathbb{K}^{n\times p}$ such that $(A)_{ik} = a_{ik}$, and $(B)_{kj} = b_{kj}$.

$$AB = \underbrace{[\ A\mathbf{b}_1\ \cdots\ A\mathbf{b}_j\ \cdots\ A\mathbf{b}_p\ ]}$$

$$\begin{bmatrix}a_{11}&a_{12}&\dots&a_{1n}\\ \vdots&&&\vdots\\ a_{i1}&a_{i2}&\dots&a_{in}\\ \vdots&&&\vdots\\ a_{m1}&a_{m2}&\dots&a_{mn}\end{bmatrix}\begin{bmatrix}b_{1j}\\b_{2j}\\ \vdots\\ b_{nj}\end{bmatrix}=\begin{bmatrix}\star_1\\ \vdots\\ \star_i\\ \vdots\\ \star_m\end{bmatrix}\longrightarrow (AB)_{ij}$$

That is,

$$(AB)_{ij} = [\ a_{i1}\quad a_{i2}\quad \dots\quad a_{in}\ ]\begin{bmatrix}b_{1j}\\b_{2j}\\ \vdots\\ b_{nj}\end{bmatrix} = \sum_k a_{ik}\, b_{kj}$$

EXAMPLE:

$$\begin{bmatrix}\boxed{2\quad 3}\\1\ -5\end{bmatrix}\begin{bmatrix}4&3&\boxed{6}\\1&-2&\boxed{3}\end{bmatrix}=\begin{bmatrix}\star&\star&\\ \star&\star&\star\end{bmatrix}$$

1st row **3**rd column $\to$ (**1** , **3**) entry

$$\begin{bmatrix}2\quad 3\\ \boxed{1\ -5}\end{bmatrix}\begin{bmatrix}\boxed{4}&3&6\\ \boxed{1}&-2&3\end{bmatrix}=\begin{bmatrix}&\star&\star&\star\\ &&\star&\star\end{bmatrix}$$

2nd row **1**st column $\to$ (**2** , **1**) entry

PROBLEM: *Find the 2nd row of AB.*

$$AB = \begin{bmatrix}2&-5&0\\-1&3&-4\\6&-8&-7\\-3&0&9\end{bmatrix}\begin{bmatrix}4&-6\\7&1\\3&2\end{bmatrix}$$

PROBLEM: *Compute*

$$\begin{bmatrix}1&-1&2\\3&0&1\end{bmatrix}\begin{bmatrix}1&1\\2&-1\\1&0\end{bmatrix}\begin{bmatrix}1&-1&0&0\\0&0&1&1\end{bmatrix}$$

• PROPERTIES:

Let A be an $(m \times n)$ matrix, and B and C matrices of appropriate dimensions:

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $\mu\,(AB) = (\mu\,A)\,B = A\,(\mu\,B) \qquad \forall\,\mu \in \mathbb{K}$
- $\mathbb{I}_m\,A = A = A\,\mathbb{I}_n$ where $\mathbb{I}_k$ is the $(k \times k)$ **identity matrix**

→ 4.3

WARNING: In general, $AB \neq BA$

EXPANSION AXIS X: $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

ROTATION 30°: $A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

1st EXPANSION + 2nd ROTATION: $AB = \begin{bmatrix} & \end{bmatrix}$

ROTATION 30°: $A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

EXPANSION AXIS X: $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

1st ROTATION + 2nd EXPANSION: $BA = \begin{bmatrix} & \end{bmatrix}$

WARNING: In general, $AB = AC \quad \not\Rightarrow \quad B = C$

ROTATION $\pi/2$: $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

PROJECTION in X: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

1st ROTATION + 2nd PROJECTION: $AB = \begin{bmatrix} & \end{bmatrix}$

REFLECTION $x+y=0$: $C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

PROJECTION in X: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

1st REFLECTION + 2nd PROJECTION: $AC = \begin{bmatrix} & \end{bmatrix}$

WARNING: In general, $AB = 0 \quad \not\Rightarrow \quad A = 0$ or $B = 0$

PROJECTION in X: $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

PROJECTION in Y: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

1st X-PROJECTION + 2nd Y-PROJECTION: $AB = \begin{bmatrix} & \end{bmatrix}$

WARNING: In general, $A^2 = 0 \quad \not\Rightarrow \quad A = 0$

$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad A^2 = \begin{bmatrix} & \end{bmatrix}$

• If two square matrices verify that $AB = BA$, we say that A and B **commute** with each other.

• The kth **power** of a matrix is defined:

$$A^k = \underbrace{A\,A\,A\cdots A}_{k \text{ times}}$$

This only makes sense if A is a _____ matrix and k is a nonnegative integer.

• For convenience, we define $A^0 = \mathbb{I}$.

PROBLEM: *Compute*

→ 4.4

Transpose of a Matrix

• The **transpose** of an $(m \times n)$ matrix A is the $(n \times m)$ matrix A^T whose columns are the rows of A.

That is,

$$(A^T)_{ij} = (A)_{ji}$$

EXAMPLE:

$$B = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix} \quad \Rightarrow \quad B^T =$$

EXAMPLE:

• A **symmetric** matrix verifies $A^T = A$.
• An **antisymmetric** matrix verifies $A^T = -A$.

PROBLEM: *Provide examples of (anti)symmetric matrices.*

• **PROPERTIES:**

Let A and B be matrices of appropriate dimensions and $\mu \in \mathbb{K}$:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(\mu\,A)^T = \mu\,(A^T)$
- $(AB)^T = B^T\,A^T$

Proof: *Let be* $A \in \mathbb{K}^{m\times n}$ *and* $B \in \mathbb{K}^{n\times q}$

$$\left((AB)^T\right)_{ij} =$$

PROBLEM: *Prove that* $(ABC)^T = C^T\,B^T\,A^T$.

→ 4.7

Conjugate Transpose of a Matrix

• The **conjugate transpose** of an $(m \times n)$ matrix A is the $(n \times m)$ matrix A^*, or A^H, whose elements verify:

$$(A^*)_{ij} = \overline{(A)_{ji}}.$$

EXAMPLE:

$$B = \begin{bmatrix} -5 & 2-i \\ i & 3 \\ 0 & 4 \end{bmatrix} \quad \Rightarrow \quad B^* =$$

$$A = [\ \mathbf{a}_1\ \ \mathbf{a}_2\ \cdots\ \mathbf{a}_n\ ] \quad \Rightarrow \quad A^* =$$

• **PROPERTIES:**

Let A and B be matrices of appropriate dimensions and $\mu \in \mathbb{K}$:

- $(A^*)^* = A$
- $(A + B)^* = A^* + B^*$
- $(\mu\, A)^* = \bar{\mu}\,(A^*)$
- $(AB)^* = B^*\, A^*$
- $A^* = A^T$ if and only if A is a real matrix.

• A **Hermitian** matrix verifies $A^* = A$.

• An **antihermitian** matrix verifies $A^* = -A$.

PROBLEM: *Provide examples of (anti)Hermitian matrices.*

→ 4.8

4.2. Inverse of a Matrix

• A square $(n \times n)$ matrix A is **invertible**, or **nonsingular**, if there exists a matrix B such that

$$AB = \mathbb{I}_n$$

• A **noninvertible** or **singular** matrix has no inverse.

EXAMPLE: *This matrix is invertible:* $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$

Because $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ verifies $AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

EXAMPLE: *This matrix is invertible:* $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

$A =$ ☐ $\Rightarrow$ $A^{-1} =$ ☐

Thus, $A^{-1} = \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

EXAMPLE: *Matrix B has no inverse and is, therefore, a singular matrix:*

$B =$ ☐ $= \begin{bmatrix} \quad \end{bmatrix}$

→ 4.9

> **Theorem 4.1.** If A is an invertible $(n \times n)$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\,\mathbf{b}$, $\forall\,\mathbf{b} \in \mathbb{K}^n$.

Proof:

∘ That $\mathbf{x} = A^{-1}\,\mathbf{b}$ is a solution $\forall\,\mathbf{b}$ can be checked by a mere substitution:

∘ As it has a solution $\forall\,\mathbf{b}$ $\Rightarrow$ A must have a pivot in every row.

A square $\Rightarrow$ No free variables $\Rightarrow$

Warning:

Theorem 4.2. Let A and B be $(n \times n)$ matrices. Then:

$$AB = \mathbb{I} \quad \Leftrightarrow \quad BA = \mathbb{I}$$

Proof: $(AB = \mathbb{I} \ \Rightarrow \ BA = \mathbb{I})$

∘ Suppose that $BA = X$

∘ Let's define $M = \mathbb{I} - X = [\, \mathbf{m}_1 \ \mathbf{m}_2 \ \cdots \ \mathbf{m}_n \,]$.

As

That is,

∘ But now,

Leading to

Theorem 4.3. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Proof:

Theorem 4.4. If exists, the inverse of a matrix is unique.

Proof: *Let A be an invertible matrix, and B a matrix such that $AB = \mathbb{I}$ (that is, $B = A^{-1}$). Suppose there exists C such that $AC = \mathbb{I}$ (in other words, suppose that A has another inverse).*

Theorem 4.5. If A is invertible, A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 4.6. If A is invertible, A^* is also invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proof:

EXAMPLE:

$$\begin{bmatrix} 1+i & 1+2i \\ -1 & -1-i \end{bmatrix} \begin{bmatrix} -1-i & -1-2i \\ 1 & 1+i \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

then, $$\begin{bmatrix} -1+i & 1 \\ -1+2i & 1-i \end{bmatrix}^{-1} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Theorem 4.7. If A and B are invertible $(n \times n)$ matrices, then AB is invertible and $(AB)^{-1} = B^{-1} A^{-1}$.

Proof:

EXAMPLE: *Consider the linear transformations:*

$A =$ *ROTATE* $\qquad B =$ *EXPAND*.

Then,

$AB = \boxed{\quad} \boxed{\quad} = \boxed{\qquad}$

(in this order!) and the inverse is

$(AB)^{-1} = \boxed{\qquad} = \boxed{\quad}^{-1} \boxed{\quad}^{-1}$

PROBLEM: *If A, B and C are nonsingular matrices of equal size, show that $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$.*

→ 4.11

• An **elementary matrix** is one that is obtained by performing one elementary row operation on an identity matrix.

EXAMPLE:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice: These matrices have a clear geometrical interpretation. They correspond to

Theorem 4.8. If an elementary row operation if performed on an $(m \times n)$ matrix A, the resulting matrix can be written as EA, where E is the $(m \times m)$ elementary matrix created by performing the same operation on $\mathbb{I}_m$.

EXAMPLE: *Consider the* (3×2) *matrix* $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

∘ $\mathbb{I} \sim E_1 \quad (r_3 \to 5\,r_3)$

$$E_1\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

∘ $\mathbb{I} \sim E_2 \quad (r_2 \leftrightarrow r_3)$

$$E_2\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

∘ $\mathbb{I} \sim E_3 \quad (r_2 \to r_2 - 4r_1)$

$$E_3\,A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$A \sim$	$A \sim$	$A \sim$

Theorem 4.9. Every elementary matrix E is invertible and its inverse E^{-1} is the elementary matrix corresponding to the row operation that transforms E back into $\mathbb{I}$.

EXAMPLE: *The matrix* E_1 *multiplies the 3rd row by five:*

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Its inverse E_1^{-1} *is the matrix that divides the 3rd row by five:*

$$E_1^{-1} = \begin{bmatrix} \quad \end{bmatrix}$$

Check: $E_1\,E_1^{-1} = \cdots = \mathbb{I}$

→ 4.12 **PROBLEM:** *Find the matrices* E_2^{-1} *and* E_3^{-1}.

Theorem 4.10. An $(n \times n)$ matrix A is invertible if and only if A is row equivalent to $\mathbb{I}_n$. In this case, any sequence of elementary row operations that transforms A into $\mathbb{I}_n$ also transforms $\mathbb{I}_n$ in A^{-1}.

Proof:

A *invertible* $\Leftrightarrow$

$\Rightarrow$

Then, $A^{-1} = E_p\, E_{p-1} \, \ldots \, E_2\, E_1$ *and, in fact,*

→ 4.14

An Algorithm for finding A^{-1}

- Construct the matrix $[\, A \;\; \mathbb{I} \,]$
- Find its reduced echelon form.
- If this matrix has the form $[\, \mathbb{I} \;\; B \,]$, then $A^{-1} = B$. Otherwise, A does not have an inverse.

EXAMPLE:

$$\left[\underbrace{\begin{matrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{matrix}}_{A} \;\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\mathbb{I}}\right] \sim \left[\underbrace{\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{matrix}} \;\; \underbrace{\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}}\right] \sim$$

$$\left[\underbrace{\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{matrix}} \;\; \underbrace{\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -4 & 1 \end{matrix}}_{\mathbb{I}}\right] \sim \left[\underbrace{\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix}} \;\; \underbrace{\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{matrix}}\right] \sim$$

$$\left[\begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{matrix}\right] \sim$$

$$\left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} \;\; \underbrace{\begin{matrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{matrix}}\right] \Rightarrow A^{-1} = \left[\begin{matrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{matrix}\right]$$

PROBLEM: *If exists, find the inverse of the matrix*

$$C = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

$$[\, C \;\; \mathbb{I} \,] = \left[\begin{matrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{matrix}\right]$$

Check: $\quad C\,C^{-1} =$

→ 4.16

Theorem 4.11. (The Square Matrix Theorem)

If $A \in \mathbb{K}^{n\times n}$, the following statements are equivalent:

1. A is an invertible matrix.
2. There exists $C \in \mathbb{K}^{n\times n}$ such that $AC = \mathbb{I}_n$.
3. There exists $D \in \mathbb{K}^{n\times n}$ such that $DA = \mathbb{I}_n$.
4. A is row equivalent to $\mathbb{I}_n$.
5. A has n pivots.
6. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
7. The columns/rows of A are linearly independent.
8. The equation $A\mathbf{x} = \mathbf{b}$ has a (unique) solution $\forall\, \mathbf{b} \in \mathbb{K}^n$.
9. The columns/rows of A span $\mathbb{K}^n$.
10. The columns/rows of A form a basis of $\mathbb{K}^n$
11. A^T is invertible.
12. A^* is invertible.
13. The linear transformation $\mathbf{x} \to A\mathbf{x}$ is bijective.
14. Col A = Row $A = \mathbb{K}^n$
15. dim Col A = dim Row $A = n$
16. rank $A = n$
17. Nul $A = \{\mathbf{0}\}$
18. dim Nul $A = 0$

- A transformation $T : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ is called **invertible** if there exists a transformation $S : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ such that

$$\left.\begin{array}{l} S(T(\mathbf{x})) = \mathbf{x} \\ T(S(\mathbf{x})) = \mathbf{x} \end{array}\right\} \quad \forall\, \mathbf{x} \in \mathbb{K}^n.$$

The transformation S is called the **inverse** of T.

Theorem 4.12. Let $T : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ be a linear transformation and A its canonical matrix. T is invertible if and only if A is nonsingular. In this case, $S(\mathbf{x}) = A^{-1}\mathbf{x}$.

→ 4.17

4.3. Partitioned (or Block) Matrices

EXAMPLE:

$$A = \left[\begin{array}{rrr|rr|r} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

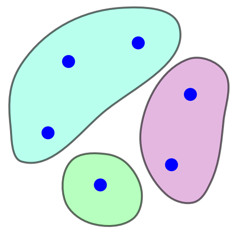
$$A = \left[\begin{array}{c|c|c} \quad & \quad & \quad \\ \hline \quad & \quad & \quad \end{array}\right] = \left[\qquad \right]$$

where

$A_{11} = [\quad]$, $A_{12} = [\quad]$, $A_{13} = [\quad]$

$A_{21} = [\quad]$, $A_{22} = [\quad]$, $A_{23} = [\quad]$

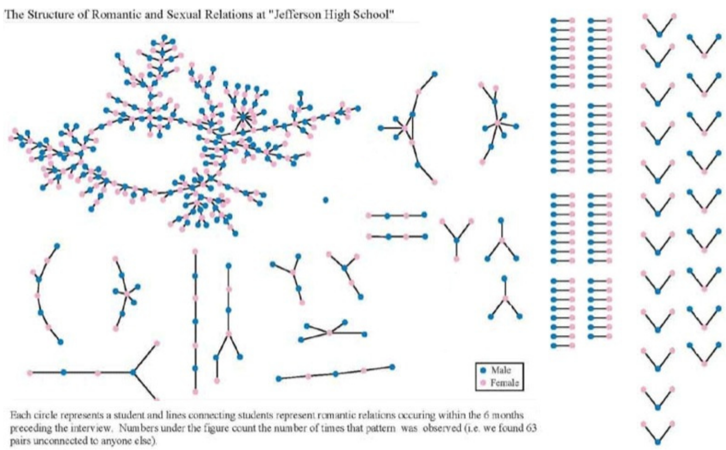
EXAMPLE: *Social web of 6 persons in 3 groups*



Adjacency Matrix

$$M = \left[\begin{array}{cc|ccc|c} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

EXAMPLE: *Jefferson High School*



EXAMPLE: *Trade share matrix between countries*

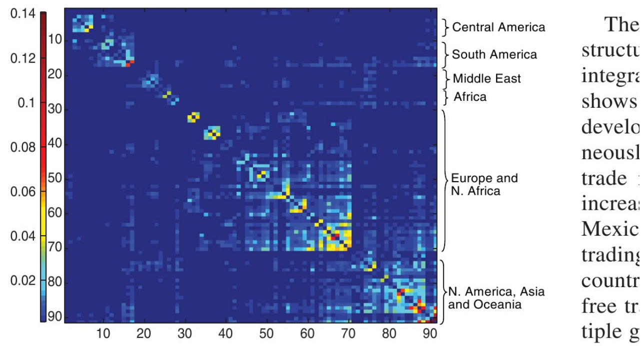

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FIG. 3 (color online). The trade share matrix $S_{ij} = M_{ij}/(\sum_{m=1}^{N} M_{im} + \sum_{n=1}^{N} M_{jn})$ after hierarchical clustering between countries in 2007. We can see clearly several modules:

• PROPERTIES:

○ **Addition:** Matrices of equal size and identical partition can be summed block by block:

$$A + B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$

○ **Scalar Multiplication:**

$$\lambda A = \lambda \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$

○ **Transpose of a matrix:**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \\ A_{13}^T & A_{23}^T \end{bmatrix} \neq \begin{bmatrix} \quad & \quad \end{bmatrix}$$

○ **Conjugate transpose of a matrix:**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \Rightarrow A^* = \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \\ A_{13}^* & A_{23}^* \end{bmatrix}$$

EXAMPLE:

$$A = \left[\begin{array}{cc|c} 2 & 0 & 8 \\ 1 & -5 & 3 \\ \hline 0 & -2 & 7 \end{array}\right] \Rightarrow A^T = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -5 & -2 \\ \hline 8 & 3 & 7 \end{array}\right]$$

∘ **Multiplication of partitioned matrices:** Two matrices A and B of respective dimensions $(m \times n)$ and $(n \times p)$ are conformable for block multiplication when the number of columns of each partition of A is equal to the number of rows of the corresponding partition of B.

$$AB = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right]$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad\qquad \end{bmatrix}$$

(**Attention:** $\qquad\qquad\qquad\qquad\qquad\qquad$)

Concentrate on the dimensions of the blocks:

$$\left[\,(3 \times 5)\,\right]\left[\,(5 \times 2)\,\right] = \begin{bmatrix} (2 \times 3) & (\qquad) \\ (\qquad) & (\qquad) \end{bmatrix} \begin{bmatrix} (\qquad) \\ (\qquad) \end{bmatrix} =$$

$$= \begin{bmatrix} (2 \times 3)(3 \times 2) + (\qquad)(\qquad) \\ (\qquad)(\qquad) + (\qquad)(\qquad) \end{bmatrix} =$$

$$= \begin{bmatrix} (2 \times 2) + (\qquad) \\ (\qquad) + (\qquad) \end{bmatrix} = \begin{bmatrix} (\qquad) \\ (\qquad) \end{bmatrix} = \left[\,(\qquad)\,\right]$$

EXAMPLE: *Let A be a block upper triangular matrix:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

Assuming that A is invertible, A_{11} is $(p \times p)$ and A_{22} is $(q \times q)$, find a formula for A^{-1}.

Call $B = A^{-1}$. Partition B in such a way that we can write:

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}.$$

The dimensions of the matrices involved are:

$$\begin{bmatrix} (p \times p) & (\quad) \\ (\quad) & (q \times q) \end{bmatrix} \begin{bmatrix} (\quad) & (\quad) \\ (\quad) & (\quad) \end{bmatrix} = \begin{bmatrix} (\quad) & (\quad) \\ (\quad) & (\quad) \end{bmatrix}.$$

The equation can be written:

$$\begin{bmatrix} \qquad\qquad\qquad\qquad\qquad\qquad \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}.$$

Equating components, we obtain:

$$\begin{array}{lcl} \text{(a)} & \qquad\qquad\qquad & = \mathbb{I} \\ \text{(b)} & & = 0 \\ \text{(c)} & & = 0 \\ \text{(d)} & & = \mathbb{I} \end{array}$$

We have to solve 4 matrix equations, which represent a linear system of $(p+q)^2$ equations with $(p+q)^2$ unknowns.

∘ (d)

∘ (c)

∘ (a)

∘ (b)

Obtaining,

$$A^{-1} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}.$$

Theorem 4.13. A block diagonal matrix is invertible if and only if each of the diagonal blocks is invertible.

Proof: *The case of two blocks follows from the above result when $A_{12} = 0$.*

$$\left[\begin{array}{c|c|c|c} C_{11} & 0 & \dots & 0 \\ \hline 0 & C_{22} & & 0 \\ \hline \vdots & & \ddots & \\ \hline 0 & 0 & & C_{nn} \end{array}\right]^{-1} = \left[\begin{array}{c|c|c|c} & 0 & \dots & 0 \\ \hline 0 & & & 0 \\ \hline \vdots & & \ddots & \\ \hline 0 & 0 & & \end{array}\right]$$

Theorem 4.14. A diagonal matrix is invertible if and only if none of its diagonal elements is zero.

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}$$

PROBLEM: *Determine under what conditions the following matrix is invertible and, in that case, find its inverse:*

$$\begin{bmatrix} \mathbb{I}_m & 0 \\ A & \mathbb{I}_n \end{bmatrix}.$$

→ 4.19

4.4. Determinants

• Given an $(m \times n)$ matrix A, we define the **minor** A_{ij} as the $((m-1) \times (n-1))$ matrix obtained by removing the ith row and the jth column of the matrix A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

• Let A be an $(n \times n)$ matrix whose entry $(A)_{ij} = a_{ij}$. We define the **determinant** of A as

$$\det A = |A| = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det A_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j},$$

where $C_{ij} = (-1)^{i+j} \det A_{ij}$ is refered to as the ij **cofactor** of A.

Theorem 4.15. The determinant of a square matrix A can be expressed as the cofactor expansion along <u>any</u> row of the matrix

$$\det A = \sum_{j=1}^{n} (-1)^{k+j}\, a_{kj}\, \det A_{kj} = \sum_{j=1}^{n} a_{kj}\, C_{kj} \quad \begin{pmatrix}\text{along the} \\ k\text{th row}\end{pmatrix}$$

WARNING:

○

○

EXAMPLE:

$$\det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

1st row:

$=$

2nd row:

$=$

Theorem 4.16. If A is an $(n \times n)$ triangular matrix, its determinant is the product of its diagonal entries.

$$\det \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ \star & a_{22} & 0 & 0 & 0 \\ \star & \star & a_{33} & 0 & 0 \\ \star & \star & \star & a_{44} & 0 \\ \star & \star & \star & \star & a_{55} \end{bmatrix} =$$

Theorem 4.17. Let A be an $(n \times n)$ matrix.
If we obtain a matrix B,

- By adding to a row of A the multiple of another row,
$$\det B = \det A.$$
- By multiplying <u>one</u> row of A by λ,
$$\det B = \lambda \det A.$$
- By interchanging <u>two</u> rows of A,
$$\det B = -\det A.$$

EXAMPLE:

$$\begin{vmatrix} -3 & 3 & -3 \\ -2 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \left| \quad \right| = \left| \quad \right| = \left| \quad \right| =$$

Theorem 4.18. Let A be a square matrix and U an echelon matrix obtained from A by adding multiples of rows and r row interchanges (but without multiplying any row by a scalar!). Then,

$$\det A = \begin{cases} 0 & \text{if } A \text{ is not invertible} \\ (-1)^r \cdot \begin{pmatrix}\text{product of} \\ \text{the pivots}\end{pmatrix} & \text{if } A \text{ is invertible} \end{cases}$$

Proof:

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Note: This would add a new statement to theorem 4.11:

19. The determinant of A is nonzero.

WARNING: In general,

$$A \sim B \quad \not\Rightarrow \quad \det A = \det B.$$

Check theorem 4.17!

WARNING: In general,

$$\det(A+B) \neq \det A + \det B.$$

EXAMPLE: *If it was true, all determinants would be zero:*

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \right)$$

Theorem 4.19. If A and B are square matrices,

$$\det(AB) = \det A \det B.$$

Theorem 4.20. If A is a square matrix,

$$|A^T| = |A| \quad \text{and} \quad |A^*| = \overline{|A|}$$

Proof:

- *For elementary matrices, it's easy to see that* $|E| = |E^T|$.
- *If we obtain an echelon form of a matrix* A*:*

Leading to

- *Now, as* U *is a triangular matrix,* $|U^T| = |U|$ *and, consequently*

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