CEU Universidad San Pablo

UNIT I – Review of statistics: Two Random Variables

Independency

Two r.v.. *X* and *Y* are independent if any couple of events *A* (related to *X*) and *B* (related to *Y*) are independent. This leads to:

$$P\left(X \le x, Y \le y\right) = P(X \le x)P(Y \le y)$$

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

Also, it can be proved that if X and Y are independent, then:

$$F_X(x \mid Y \le y) = F_X(x) \qquad f_X(x \mid Y \le y) = f_X(x)$$

Independency

Example: Given the joint p.d.f. below, check if X and Y are independent

$$f_{XY}(x,y) = \frac{1}{12} u(x) u(y) e^{-(x/4)} e^{-(y/3)}$$

$$f_X(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = (1/4) u(x) e^{-x/4}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = (1/3) u(y) e^{-y/3}$$

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

X and Y are **independent**

Expectation of a function of multiple random variables

- The expectation of a function of multiple random variables is

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{XY}(x,y)dxdy$$

- If the r.v. are independent, then

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Moment about the origin of two random variables

- The joint moment is defined as

$$m_{nk} = E[X^n Y^k]$$

- Moment m_{11} is especially interesting and it is called the correlation of X and Y

$$R_{XY} = m_{11} = E[XY]$$

- Two r.v. are uncorrelated if

$$R_{XY} = E[X]E[Y]$$

Note that is X and Y are **independent** this means that they are always **uncorrelated**

However, the opposite does not have to be necessarily true

- Two r.v. are **orthogonal** if

Central moments of two random variables

- The joint central moment is defined as

$$\mu_{nk} = E[(X - \overline{X})^n (Y - \overline{Y})^k]$$

- Note that the joint central moment can be used to obtain the variance

$$\mu_{20} = E[(X - \overline{X})^2] = \sigma_X^2$$

$$\mu_{02} = E[(Y - \overline{Y})^2] = \sigma_Y^2$$

- Moment μ11 is especially important, and it is called the **covariance** of X and Y

$$C_{XY} = \mu_{11} = E[(X - \overline{X})(Y - \overline{Y})]$$

- The covariance is related to the **correlation** and the means of X and Y:

Covariance, correlation and orthogonality

- The covariance is also related with the concepts of correlation and orthogonality
- If two r.v.. X and Y are uncorrelated then

$$C_{XY} = R_{XY} - E[X]E[Y] = E[X]E[Y] - E[X]E[Y] = 0$$

- If two r.v. are orthogonal and the mean of any of the two is zero

$$C_{XY} = R_{XY} - E[X]E[Y] = 0 - E[X]E[Y] = 0$$

Orthogonal + E[X]=0 or E[Y]=0

Implies X and Y are **uncorrelated**

- The **correlation** coefficient of X and Y is defined as

$$\rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y}, \quad -1 \leq \rho_{XY} \leq 1$$

Variance and correlation

The variance of the sum of several uncorrelated r.v. is the sum of the variances of each r.v.

$$Var(\sum X_i) = \sum Var(X_i)$$

What if they are independent?