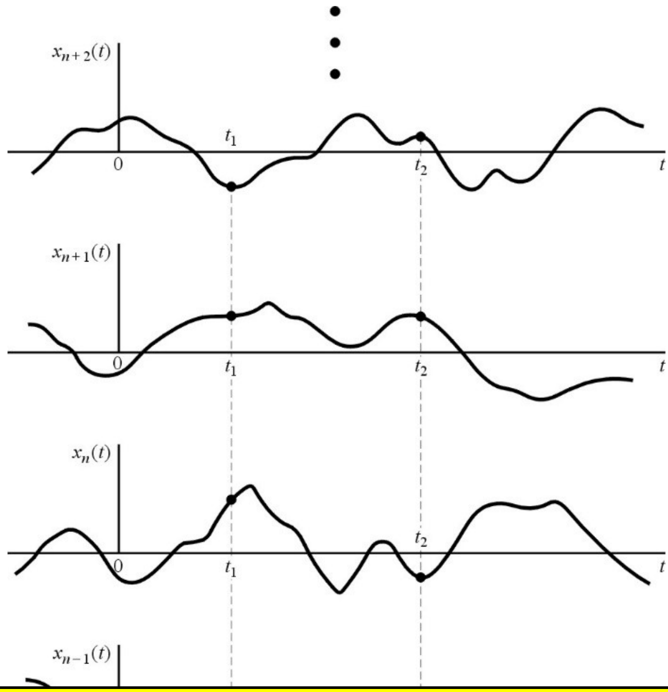
CEU
Universidad
San Pablo

# UNIT 2 – Part I: Random Processes
# Temporal Characteristics

# Random Processes

Given a function x*(t,s)* that relates the elements in the sample space *s* and time *t*

- A random process *X(t,s)* is a family or *ensemble* of *x(t,s)* functions
- Each function is called a sample or ensemble function
- $X_i=X(t_i,s)$ is a random variable (time is fixed to $t_i$)
- *The statistics of* $X_i$ are the statistics of the process at time $t=t_i$



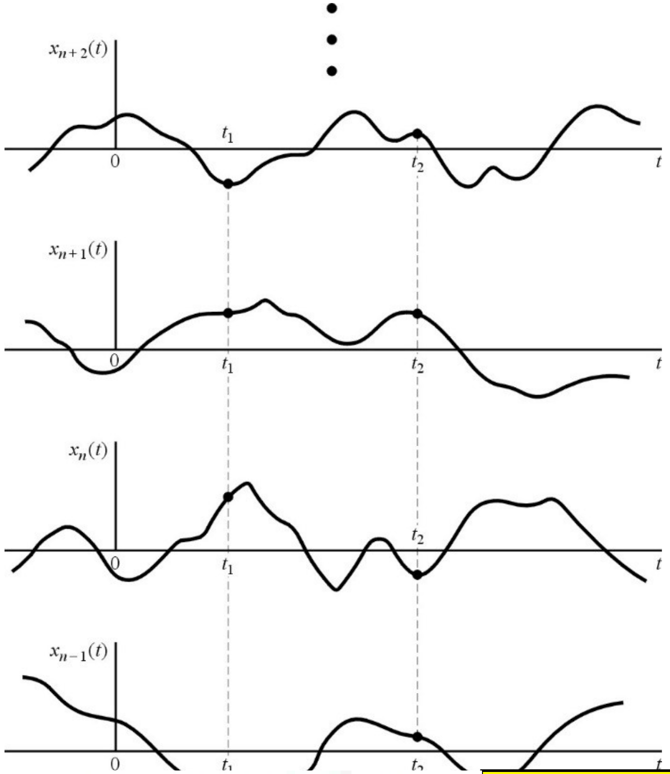

EXAMPLE

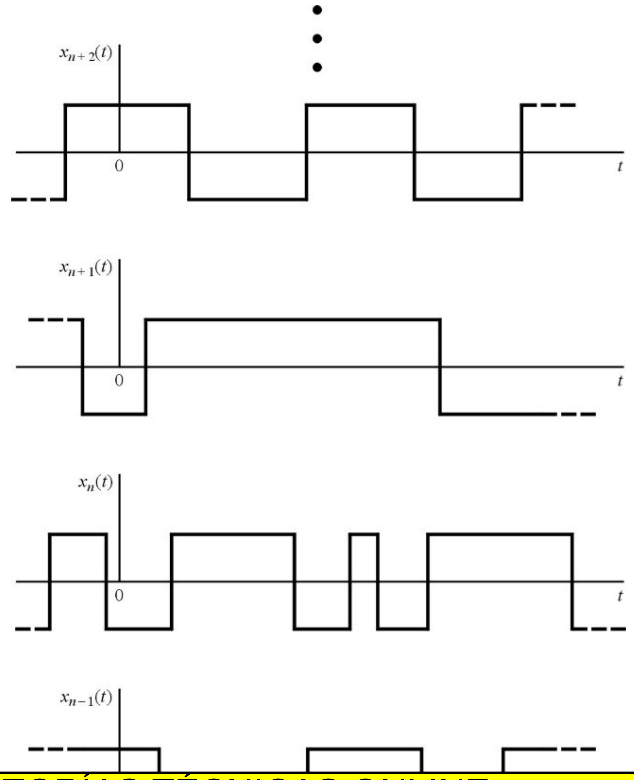
$$X(t) = a\cos(\omega_0 t + \varphi),$$

# Random Processes

Classification:

**Continuous** | **Discrete**



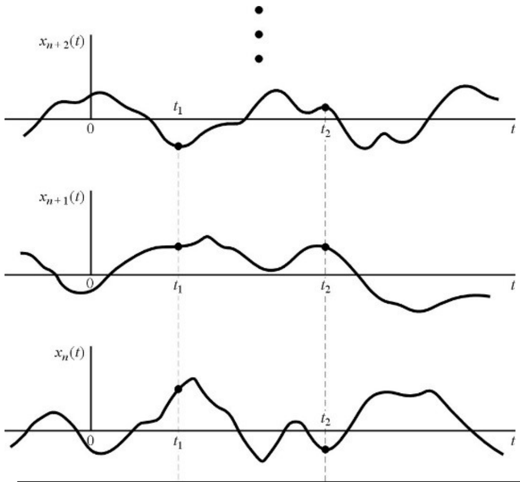
# Random Processes

Classification:

- Deterministic: Future values of the process can be predicted from past values

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0,2\pi)$$

- Undeterministic: It is not possible to predict future values



# Distribution Function

- For a given time $t_1$ the distribution function is defined as

$$F_x(x_1;t_1) = P\{X(t_1) \le x_1\}$$

- For two random variables $X_1=X(t_1)$ and $X_2=X(t_2)$

$$F_x(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

- And for *n* random variables

$$F_x(x_1,\ldots, x_N; t_1,\ldots, t_N) = P\{X(t_1) \le x_1,\ldots, X(t_N) \le x_N\}$$

# Density Function

- The probability density functions for one, two and *n* r.v. are

$$f_X(x_1;t_1) = \frac{dF_X(x_1;t_1)}{dx}$$

$$f_X(x_1,x_2;t_1,t_2) = \frac{\partial^2 F_X(x_1,x_2;t_1,t_2)}{\partial x_1 \partial x_2}$$

$$f_X(x_1,\ldots,x_N;t_1,\ldots,t_N) = \frac{\partial^N F_X(x_1,\ldots,x_N;t_1,\ldots,t_N)}{\partial x_1 \ldots \partial x_N}$$

# Independency

- Two processes *X* and *Y* are independent if

$$f_{XY}(x_1,\ldots,x_N,y_1,\ldots,y_N;t_1,\ldots,t_N,t'_1,\ldots,t'_N) =$$
$$f_X(x_1,\ldots,x_N;t_1,\ldots,t_N)\,f_Y(y_1,\ldots,y_N;t'_1,\ldots,t'_N)$$

# First-order stationary processes

- A random process is *stationary to order one* if the p.d.f. does not change with a shift in time origin

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta)$$

- This implies that

$$E[X(t_1)] = \overline{X} = \text{constant}$$

# Second-order stationarity

- A random process is *stationary to order two* if the p.d.f. satisfies

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

- This implies that $R_{XX}(t_1,t_2)$, called ***autocorrelation***, is a function of $\tau = t_2 - t_1$

$$R_{XX}(t_1, t_2) = R_{XX}(t_1, t_1 + \tau) = E[X(t_1)X(t_1 + \tau)] = R_{XX}(\tau)$$

The correlation applied to $X(t_1)$ and $X(t_2)$ is called the **autocorrelation**

# Second-order stationarity

- EXAMPLE: Show that $X(t) = A\cos(\omega_0 t + \Theta)$ is stationary to order two.

  *A* and $\omega_0$ are constants and $\Theta \sim U(0,2\pi)$

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$$R_{XX}(t_1, t_1+\tau) = \frac{A^2}{2}\cos(\omega_0 \tau) = R_{XX}(\tau)$$

# Second-order stationarity

- EXAMPLE: Check if $X(t) = A\cos(\omega_0 t + \Theta)$ is first- and second-order stationary.

  *A* and $\omega_0$ are constants and $\Theta \sim U(0, \pi)$

$$E[X(t)] = \frac{A}{2\pi}\sin(\omega_0 t)$$

$$R_{XX}(t_1, t_1 + \tau) = \frac{A^2}{2}\cos(\omega_0 \tau) = R_{XX}(\tau)$$

# Wide-sense and strict-sense stationarity

- If a process *X* is stationary to orders one and two it is said to be **wide-sense stationary** (w.s.s.)

$$E[X(t_1)] = \text{constant}$$

$$R_{XX}(t_1, t_2) = R_{XX}(\tau)$$

- Two w.s.s. processes *X* and *Y* are jointly wide-sense stationary if they are w.s.s. and

$$R_{XY}(t_1, t_2) = R_{XY}(\tau)$$

Now, it is called the **cross-correlation**

# Wide-sense and strict-sense stationarity

- CHALLENGE: Show graphically that $X(t) = A\cos(\omega_0 t + \varphi)$ is not stationary.

  $\omega_0$ and $\phi$ are constants and $A \sim U(0,1)$

# Ergodicity

- The time average of a quantity is defined as

$$A[\cdot] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\cdot] dt$$

- The **time average** of a sample function **x(t)** is

$$\bar{x} = A[x(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt$$

- The **time autocorrelation** function is

# Ergodicity

- A wide-sense stationary process X is **ergodic in the mean** if

$$E[X] = \overline{X} = A[x(t)] = \overline{x}$$

- A wide-sense stationary process X is **ergodic in the autocorrelation** if

$$R_{XX}(\tau) = \Re xx(\tau)$$

Computing the time averages of a **single** sample function gives us the statistics of the process

- Two jointly wide-sense processes X and Y are **jointly ergodic** if they are individually ergodic and

# Correlation, cross-correlation and covariance

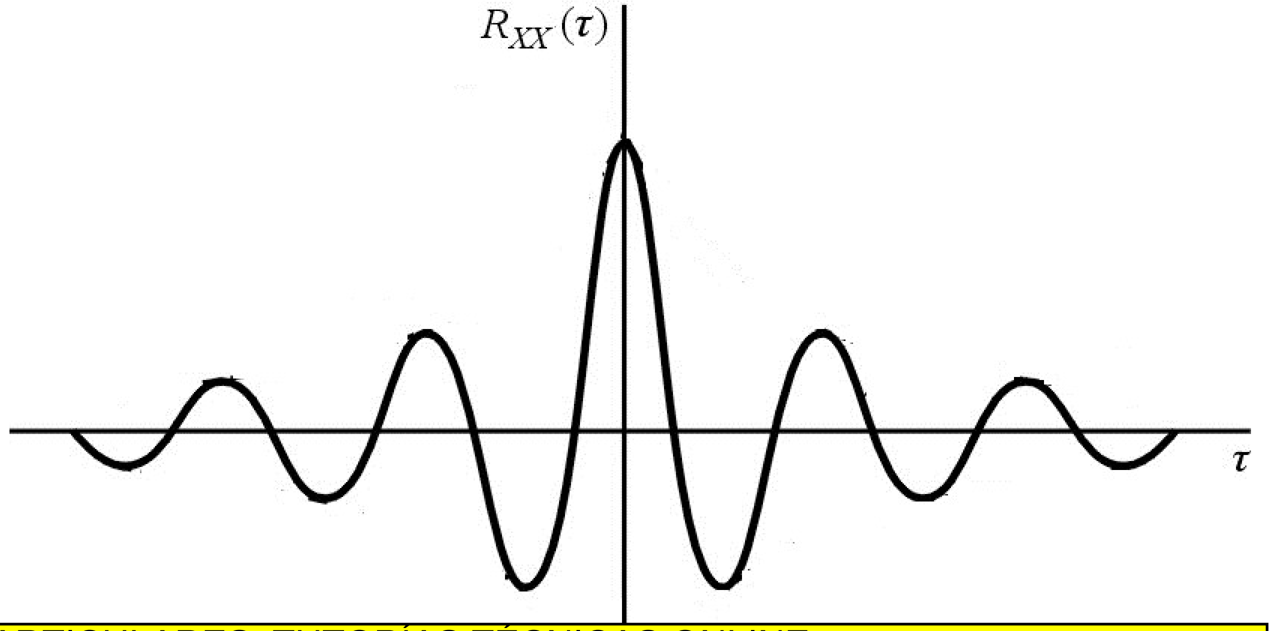
- Some properties of the **autocorrelation** of a **w.s.s.** process

1. $|R_{XX}(\tau)| \leq R_{XX}(0)$

2. $R_{XX}(-\tau) = R_{XX}(\tau)$

3. $R_{XX}(0) = E[X^2(t)]$

**Power** of the process



# Correlation, cross-correlation and covariance

- Some properties of the **cross-correlation** of **w.s.s.** processes

1. $R_{XY}(-\tau) = R_{YX}(\tau)$

2. $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

- Two processes are **orthogonal** if

$$R_{XY}(t, t+\tau) = 0$$

- Two processes are **uncorrelated** if

# Correlation, cross-correlation and covariance

- EXAMPLE: Given two w.s.s. random processes

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
$$Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$$

where *A* and *B* are uncorrelated, zero-mean r.v. with the same variance, check if X and Y are jointly wide-sense stationary

$$R_{XY}(t_1, t_1+\tau) = -\sigma^2 \sin(\omega_0 \tau) = R_{XY}(\tau)$$

# Correlation, cross-correlation and covariance

- The **autocovariance** of X is

$$C_{XX}(t,t+\tau) = E\big[\big(X(t)-E[X(t)]\big)\big(X(t+\tau)-E[X(t+\tau)]\big)\big]$$
$$= R_{XX}(t,t+\tau) - E[X(t)]E[X(t+\tau)]$$

If X is at least w.s.s. $C_{XX}(\tau) = R_{XX}(\tau) - (\overline{X})^2$

- The **cross-covariance** of X and Y is

$$C_{XY}(t,t+\tau) = E\big[\big(X(t)-E[X(t)]\big)\big(Y(t+\tau)-E[Y(t+\tau)]\big)\big]$$
$$= R_{XY}(t,t+\tau) - E[X(t)]E[Y(t+\tau)]$$

If X and Y are at least **jointly w.s.s.**

# Correlation, cross-correlation and covariance

- The **variance** of X can be computed from the **autocovariance**

$$\sigma_X^2 = E\left[\left(X(t) - E[X(t)]\right)^2\right] = C_{XX}(t,t) \xrightarrow{w.s.s.} C_{XX}(0) = R_{XX}(0) - \overline{X}^2$$

- Two processes X and Y are **uncorrelated** if

$$C_{XY}(t, t+\tau) = 0$$

$$R_{XY}(t, t+\tau) = E[X(t)]E[Y(t+\tau)]$$

# SUMMARY

- Random processes
- Distribution and density functions
- Stationarity and ergodicity
- Auto- and Cross-Correlation
- Auto- and Cross-covariance
- Orthogonality
- Independence
- Correlation