The Finite Element Method

Section 1

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Course Aims and Objectives

- formulate the weak Galerkin approximation for PDEs in one or more spatial dimensions
- define Dirichlet and Neumann boundary conditions and explain how to enforce them within the Galerkin finite element framework
- show how an elemental decomposition and numerical integration may be used to construct the global matrix system arising from the Galerkin finite element technique
- recognise the applicability of the finite element method across many fields of engineering
- extend the developed method to the solution of time-dependent (parabolic) problems

A brief history of the Finite Element Method

The finite element method was initially developed from matrix methods of structural analysis.

- FEM has its beginnings in WW II aerospace
- Early pioneers: Argyris (this department); Clough (coined the term 'finite element' in 1960)
- FEM originally confined to structural analysis — aerospace and civil engineering
- Rigorous mathematical foundation developed in the 1970s — application to CFD, heat transfer, finance, ...

The disparate origins mean that multiple words are often used to express the same concept.

Course Approach

We will focus on the FEM as a tool to solve PDEs rather than on specific fields of engineering.

- 10 Lectures and 1 Tutorial
- The lectures will combine both slides and the visualizer
- Assessment is 100% by coursework

The coursework will require approximately 1 week of work.

Boundary Conditions

We start with some important definitions.

- A **Dirichlet** boundary condition specifies the value of a solution on the boundary of a domain.
- A **Neumann** boundary condition specifies the value of the derivative of a solution on the boundary of a domain.
- An **Essential** boundary condition is one which must be specified in the integral statement of a problem.
- A **Natural** boundary condition is one which arises as a natural consequence of the integral statement of a problem.

A boundary condition can be **mixed**, i.e. a combination of a Dirichlet and Neuman condition

Strong and Weak forms

- The **Strong form** of an equation, denoted (S), is the familiar form of a PDE in which differentiation is carried out of the dependent variables, and both Dirichlet and Neumann boundary conditions are specified as required.
- The **Weak form** of an equation, denoted (W), is an integral form of a PDE, specifically:

> "... *a weighted integral statement of a differential equation in which the differentiation is transferred from the dependent variable to the weight function such that all natural boundary conditions of a problem are included in the integral statement.*"
>
> — Reddy

Symmetric and Bilinear Functions

We denote $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ to be symmetric bilinear forms.

A symmetric function has the property:

$$\begin{aligned} a(u,v) &= a(v,u) \\ (u,v) &= (v,u) \end{aligned}$$

A bilinear function has the property:

$$\begin{aligned} a(c_1 u + c_2 v, w) &= c_1 a(u,w) + c_2 a(v,w) \\ (c_1 u + c_2 v, w) &= c_1 (u,w) + c_2 (v,w) \end{aligned}$$

The Ritz Method

We require the solution u to the variational equation

$$B(w, u) = l(w)$$

We approximate u as the weighted sum of a series of n linearly-independent basis functions

$$u_n(x) = \sum_{j=1}^{n} d_j \phi_j(x) + \phi_0(x) \approx u$$

in which d_j are 'Ritz coefficients' chosen such that for each $w = \phi_i$ $(i = 1 \ldots n)$, $B(w, u) = l(w)$ holds for n different choices of w, and ϕ_j $(j = 0 \ldots n)$ are approximation functions chosen such that u_n satisfies the essential boundary conditions.

The Ritz Method

Substituting the approximate solution into the variational equation gives

$$B(\phi_i, \sum_j d_j \phi_j + \phi_0) = l(\phi_i)$$

Invoking the bilinearity of $B(\cdot, \cdot)$ allows us to re-express this as

$$\sum_j B(\phi_i, \phi_j) d_j = l(\phi_i) - B(\phi_i, \phi_0)$$

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This may be more concisely expressed as the matrix equation

$$[K_{ij}]\{d_j\} = \{F_i\}$$

Coefficients d_j are obtained by inversion of the matrix K_{ij}. The approximation functions ϕ_j are known, and hence the approximate solution $u_n \approx u$ may be determined.

Weighted Residual Methods

A generalization of the Ritz method and applicable to a much broader class of problems. We consider the general boundary value problem

$$A(u) = f \quad \text{on } \Omega$$

in which $A(\cdot)$ is a linear or nonlinear operator, and Ω is the domain over which the problem is specified.

Visualizer

Weighted residuals and the Petrov-Galerkin method

The Galerkin Method

The weight functions are identical to the approximations functions

$$\psi_i = \phi_i$$

As with the Petrov-Galerkin method, the weighted residual integral may be represented as a matrix equation

$$[K_{ij}]\{d_j\} = \{f_i\}$$

but now the matrix $[K_{ij}]$ is symmetric which has useful implications for the efficiency of solution.

If it is possible to transfer differentiation from the dependent variables to the weight functions via integration by parts to obtain the weak form of the problem, then the Galerkin method is identical to the Ritz method and is used synonymously.

Summary

Key points of this section:

- We have introduced the weighted residuals method for the solution of the general BVP
$$A(u) = f \quad \text{on } \Omega$$
- This has been shown to be equivalent to the Galerkin method in special cases
- The Galerkin method forms the basis of the F.E.M.

In the following section we will apply this approach to a simple 1-D problem and introduce the concept of ‘finite element’