Exercises for Differential calculus in several variables. Bachelor Degree Biomedical Engineering
Universidad Carlos III de Madrid. Departamento de Matemáticas

Chapter 3.2 Change of variable

Problem 1. Use a linear transformation to compute the double integral

$$\int_S (x-y)^2 \sin^2(x+y)\, dxdy,$$

where S is the parallelogram with vertices $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Solution: $\pi^4/3$.

Problem 2. Consider the map $\begin{cases} x = u+v \\ y = v-u^2 \end{cases}$. Compute:

i) The Jacobian matrix for the transformation $JT(u, v)$;

ii) The image S in the xy-plane of the triangle T in the UV-plane of vertices $(0,0)$, $(2,0)$ and $(0,2)$;

iii) The area of S;

iv) The integral $\int_S (x-y+1)^{-2} dxdy$.

Solution: $i)$ $1+2u$; $iii)$ $14/3$; $iv)$ $2+(\pi - 6\arctan(5/\sqrt{3}))\sqrt{3}/9$.

Problem 3. Compute the double integral $\int\int_D \log(x^2+y^2)\, dx\, dy$ where D is the region in the first quadrant defined by the curves $x^2+y^2=1$ and $x^2+y^2=4$.

Solution: $2\pi\left(\log 2 - \frac{3}{8}\right)$.

Problem 4. Compute the integral of the function

$$f(x,y) = \frac{y^4}{b^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(1+\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)} + xy^2$$

on the region $D = \left\{ \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \right\}$, where a and b are positive constants.

Solution: $3\pi ab(1-\log 2)/8$.

Problem 5. Compute the integral of the function

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}}\, e^{\sqrt{x^2+y^2}}$$

on the region $E = \{\, x^2+(y-1)^2 \le 1 \,\}$ and $H = \{\, x^2+(y-1)^2 \le 1,\ x \ge 0 \,\}$.

Solution: $\int_E f = 0$, $\int_H f = 2$.

Problem 6. Compute the integral of the function $h(x,y) = \frac{\sqrt{2y^2+x^2}}{y}$ on the region $R = \{(x,y) \in \mathbb{R}^2 \,/\, x^2 + (y-1)^2 \leq 1,\ x \geq 0\}$.

Solution: $\int_R h = 1 + \pi/2$.

Problem 7. Compute the integral $\int_S \frac{x\,dx\,dy}{4x^2+y^2}$, where S is the region in the first quadrant defined by the lines $x = 0$, $y = 0$ and the ellipses $4x^2 + y^2 = 16,\ 4x^2 + y^2 = 1$.

Solution: $3/4$.

Problem 8. If R is the region defined by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$, compute the integrals:

i) $\int_R \sqrt{x^2+y^2+z^2}\, dxdydz.$

ii) $\int_R \sqrt{9-x^2-y^2}\, dxdydz.$

iii) $\int_R z\, e^{x^2+y^2+z^2}\, dxdydz.$

Solution: $i)$ $27\pi(2\sqrt{2}-1)/2$; $ii)$ $54\pi - 81\pi^2/8$ $iii)$ $\pi(e^9-1)^2/4$.

Problem 9. Compute $\int_W f(x,y,z)\, dxdydz$, where $f(x,y,z) = e^{-(x^2+y^2+z^2)^{3/2}}$ and W is the region below the sphere $x^2+y^2+z^2 = 9$ and above the cone $z = \sqrt{x^2+y^2}$.

Solution: $\pi(2-\sqrt{2})(1-e^{-27})/3$.
