Chapter 4.3 Surface integrals

Problem 1. Compute the area of the following surfaces:

i) A sphere of radius R;

ii) A circular cone parametrized by $\mathbf{r}(u, v) = (u \cos v, u \sin v, u)$, where $0 \le u \le a$ and $0 \le v \le 2\pi$.

iii) A piece of the paraboloid $z = x^2 + y^2$ which lies within the cylinder $x^2 + y^2 = a^2$;

iv) A piece of the cylinder $x^2 + z^2 = 16$ bounded by the cylinder $x^2 + y^2 = 16$.

Solution: $i)$ $4\pi R^2$; $ii)$ $\pi a^2\sqrt{2}$; $iii)$ $\pi((1 + 4a^2)^{3/2} - 1)/6$; $iv)$ 128.

Problem 2. Find the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ lying outside the cylinders $x^2 + y^2 = \pm ax$.

Solution: $8a^2$.

Problem 3. i) Deduce the formula of the area of a surface of revolution obtained by rotating the graph of the function $y = f(x),\ 0 < a \le x \le b$, around the vertical axis:

$$A = 2\pi \int_a^b x\sqrt{1 + (f'(x))^2}\, dx\,,$$

for the parametrization $\mathbf{s}(r, \theta) = (r \cos\theta, r \sin\theta, f(r))$, where $a \le r \le b$ and $0 \le \theta \le 2\pi$.

ii) Give the area of the surface of the torus obtained by rotating the graph $(x-R)^2+y^2 = c^2,\ 0 < c < R$.

iii) Give the corresponding parametrization for an analogous formula in the case where the graph $y = f(x),\ a \le qx \le b$, is rotated along the horizontal axis.

Solution: $ii)$ $4\pi^2 Rc$; $iii)$ $\mathbf{s}(x, \theta) = (x, f(x)\cos\theta, f(x)\sin\theta))$.

Problem 4. Consider a subset of $\mathbb{R}^3$ given by $W = \{1 \le z \le (x^2 + y^2)^{-1/2}\}$. Show that the volume of W is finite and that its boundary has infinite area.

Solution: $V = \pi$.