Exercises for Differential calculus in several variables. Bachelor Degree Biomedical Engineering

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Chapter 4.4 Theorems of Green, Stokes and Gauss

Problem 1. Compute in two ways $\int_\gamma (5 - xy - y^2)dx - (2xy - x^2)dy$, where γ is the square with vertices (0,0), (1,0), (1,1) y (0,1): applying directly the definition and using Green's theorem.

Solution: $3/2$.

Problem 2. Let f be a differentiable function in $\mathbb{R}$ and consider

$$P(x, y) = e^{x^2} - \frac{y}{3 + e^{xy}}, \quad Q(x, y) = f(y) \ .$$

If γ is the boundary of the square $[0, 1] \times [0, 1]$ walked on positively, compute $\int_\gamma P dx + Q dy$.

Solution: $(1 - \log(e + 3) + \log 4)/3$.

Problem 3. Consider the functions $P(x, y) = y/(x^2 + y^2)$ and $Q(x, y) = -x/(x^2 + y^2)$. Let $C = \partial D$ be a closed, piecewise regular curve that does not pass through the origin.

i) Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

ii) If $(0, 0) \in D$, prove that $\int_C P\, dx + Q\, dy = \pm 2\pi$.

iii) If $(0, 0) \notin D$, compute $\int_C P\, dx + Q\, dy$.

Solution: $iii)$ 0.

Problem 4. Compute $\displaystyle\int_\gamma \frac{-y\, dx + (x - 1)\, dy}{(x - 1)^2 + y^2}$, where γ is a closed, simple, piecewise regular curve containing the point $(1, 0)$ in its interior.

Solution: $\pm 2\pi$ depending on the orientation of γ.

Problem 5. i) Let A be the area of the region D, bounded by a closed, simple, piecewise regular curve C which is positively orientated. Show that in Cartesian coordinates

$$A = \frac{1}{2} \int_C -y\, dx + x\, dy \,,$$

and, on the other hand, in polar coordinates

$$A = \frac{1}{2} \int_C r^2(\theta)\, d\theta.$$

ii) Compute the area contained in the loop parametrized by $\mathbf{s}(t) = (t^2 - 1, t^3 - t)$.

iii) Compute the area of the cardioid given in polar coordinates by $r(\theta) = a(1 - \cos\theta)$, $(0 \leq \theta \leq 2\pi)$.

Solution: $ii)$ $8/15$; $iii)$ $\frac{3\pi a^2}{2}$.

Problem 6. i) Compute $\int_D (x + 2y)dxdy$, where D is the region bounded by the interval $[0, 2\pi]$ and the arc of the cycloid $x = t - \sin t,\ y = 1 - \cos t$, for $0 \leq t \leq 2\pi$.

ii) Compute $\int_D xy^2 dxdy$, where D is the region bounded by the astroid $x = \cos^3 t,\ y = \sin^3 t$, $0 \leq t \leq \pi/2$ and the two axis.

iii) Compute $\int_D y^2 dxdy$, where D is region bounded by the curve $x = a(t - \sin^2 t),\ y = a\sin^2 t$, $0 \leq t \leq \pi$, and the line connecting the end points.

Solution: $i)$ $-2\pi(3\pi + 2)$; $ii)$ $8/2145$; $iii)$ $\frac{5}{48}\pi a^4$.

Problem 7. Use Stoke's theorem to compute the following integrals $\int_S \text{curl}\,\mathbf{F}$, where the orientation of S is given by the unit outer normal to S:

i) $\mathbf{F}(x, y, z) = (x^2y^2, yz, xy)$ and S is the paraboloid $z = a^2 - x^2 - y^2,\ z \geq 0$.

ii) $\mathbf{F}(x, y, z) = ((1 - z)y, ze^x, x \sin z)$ and S is the upper semi-sphere of radius a.

iii) $\mathbf{F}(x, y, z) = (x^3 + z^3, e^x + y + z, x^3 + y^3)$ and $S = \{\, x^2 + y^2 + z^2 = 1,\ y \geq 0\,\}$.

Solution: $i)$ 0; $ii)$ $-\pi a^2$; $iii)$ 0.

Problem 8. Consider the vector field $\mathbf{F}(x, y, z) = \Big(y\,,\ x^2\,,\ (x^2+y^4)^{3/2}\ \sin(e^{\sqrt{xyz}})\Big)$. Compute $\int_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ is the unit inner normal of the semi-ellipsoid

$$S = \{(x, y.z) :\ 4\,x^2 + 9\,y^2 + 36\,z^2 = 36\,,\ z \geq 0\}\,.$$

Solution: 6π.

Problem 9. Consider the vector field $\mathbf{F}(x, y, z) = (2y, 3z, x)$ and the triangle of vertices $A(0, 0, 0)$, $B(0, 2, 0)$ and $C(1, 1, 1)$ which we denote by T.

i) Choose an orientation for the surface of the triangle T and the corresponding induced orientation for its boundary.

ii) Compute the path integral of the field $\mathbf{F}$ along the boundary of T.

Solution: $i)$ $\mathbf{n} = (1, 0, -1)$; the boundary is traversed from A to B, from B to C and from C to A; $ii)$ -1.

Problem 10. Consider the vector valued function $\mathbf{F}(x, y, z) = (y \sin(x^2 + y^2), -x \sin(x^2 + y^2), z(3 - 2y))$ and the region $W = \{(x, y, z) \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 \leq 1, z \geq 0\}$. Compute $\int_{\partial W} \mathbf{F}$.

Solution: 2π.

Problem 11. Verify Stokes' theorem in the following cases:

i) $\mathbf{F}(x,y,z) = (y^2, xy, xz)$, over the paraboloid $z = a^2 - x^2 - y^2,\ z \geq 0$.

ii) $\mathbf{F}(x,y,z) = (-y^3, x^3, z^3)$ over $S = \{\, z = y,\ y \geq 0,\ x^2 + y^2 \leq 1 \,\}$.

Solution: $i)$ 0; $ii)$ $3\pi/4$.

Problem 12. A vector field on $\mathbb{R}^3$ is given by $\mathbf{F}(x,y,z) = (P_1(x,y) + P_2(x,z), x + Q(y,z), R(x,y,z))$, with $P_1, P_2, Q, R \in \mathcal{C}^2(\mathbb{R}^3)$. If Γ_h is the section of the cylinder $x^2 + y^2 = 1$ at high h, show that $\int_{\Gamma_h} \mathbf{F}$ is independent of h.

Problem 13. Compute the integral $\int_S \mathbf{F}$, where

i) $\mathbf{F}(x,y,z) = (18z, -12, 3y)$ and S the region of the plane $2x + 3y + 6z = 12$ in the first octant.

ii) $\mathbf{F}(x,y,z) = (x^3, x^2y, x^2z)$ and S is the closed surface bounding the cylinder $x^2+y^2 = a^2,\ 0 \leq z \leq b$, including the upper and lower covers.

iii) $\mathbf{F}(x,y,z) = (4xz, -y^2, yz)$ and S is the surface bounding the cube $0 \leq x,\ y,\ z \leq 1$.

iv) $\mathbf{F}(x,y,z) = (x,y,z)$ and S is a bounded simple surface.

Solution: $i)$ 24; $ii)$ $5\pi a^4 b/4$; $iii)$ $3/2$; $iv)$ $3|\Omega|$, where $S = \partial\Omega$.

Problem 14. Let S be the square of vertices $(0,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(0,1,1)$ (oriented with the upper unit normal, i.e. positive first coordinate). Consider also the vector field

$$\mathbf{F}(x,y,z) = (xy^2, 2y^2z, 3z^2x).$$

Compute $\int_S \mathbf{curl}\,\mathbf{F} \cdot \mathbf{n}$ in two different ways (using Stokes' theorem).

Solution: $-2/3$.

Problem 15. Compute the flux of the vector field $\mathbf{F}(x,y,z) = (y^2, yz, xz)$ across the surface of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$, with orientation given by the unit outer normal to the surface.

Solution: $1/12$..

Problem 16. Assume that the temperature in $\mathbb{R}^3$ is proportional to the square of the distance to the vertical axis and consider the region $V = \{\, (x,y,z) \in \mathbb{R}^3 \,:\, x^2 + y^2 \leq 2z,\ z \leq 2 \,\}$.

i) Compute the volume of V.

ii) Compute the mean temperature on V.

iii) Compute the (outward) flux of the gradient of the temperature across ∂V.

Solution: $i)$ $16\pi/3$; $ii)$ α (the constant of proportionality); $iii)$ $32\alpha\pi$.

Problem 17. Consider S the sphere of radius a oriented with respect to its outer normal and let $\mathbf{F}(x,y,z) = (\sin yz + e^z, x\cos z + \log(1+x^2+z^2), e^{x^2+y^2+z^2})$ be a vector field. Compute $\int_S \mathbf{F}\cdot\mathbf{n}$.

Solution: 0.

Problem 18. Consider the union $S = S_1 \cup S_2$, where S_1 and S_2 are the surfaces

$$S_1 = \{\, x^2+y^2 = 1,\ 0 \le z \le 1 \,\} \qquad S_2 = \{\, x^2+y^2+(z-1)^2 = 1,\ z \ge 1 \,\},$$

and let $\mathbf{F}(x,y,z) = (zx + z^2y + x, z^3yx + y, z^4x^2)$ be a vector field.

i) Compute $\int_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}$ using Stokes' theorem.

1. Compute the same integral using Gauss' theorem.

Solution: 0.

Problem 19. Consider the differentiable function $h : \mathbb{R}^2 \to \mathbb{R}$. Compute $\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ is the unit normal inner with respect to $\partial\Omega$, and

$$\mathbf{F}(x,y,z) = \left(e^{y^2+z^2} + \int_0^x \frac{e^{t^2+y^2}}{\sqrt{t^2+y^2}}\,dt,\ \sin(x^2+e^z),\ h(x,y) \right),$$

$$\Omega = \{\, (x,\ y,\ z) \in \mathbb{R}^3 \ :\ x^2+y^2 \le 1,\ 0 \le z \le \sqrt{x^2+y^2},\ x \ge 0,\ y \ge 0 \,\}.$$

Solution: $\frac{\pi}{4}(1-e)$.

Problem 20. Consider the vector field

$$\mathbf{F}(x,y,z) = \left(y\,e^z\,,\ \int_0^x e^{-t^2+\cos z} dt\,,\ z(x^2+y^2) \right).$$

Compute $\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ is the normal outer to the boundary of the region

$$\Omega = \{(x,y,z) :\ x^2+y^2+z^2 < a^2\,,\ x^2+y^2 < z^2\}\,.$$

Solution: $(8-5\sqrt{2})\pi a^5/15$.

Problem 21. Consider the surface S given by

$$S = \left\{ (x,y,z) \mid z = 2 - \frac{1}{2}(x^2+y^2)\ ,\ z \ge 0 \right\}$$

(i) Sketch the surface S and the curve c (lying in the plane $z=0$) given as the boundary of S, i.e. $c = \partial S$.

(ii) Parametrize the curve c and use Stokes' theorem to compute the surface integral

$$I = \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

of the vector field $\mathbf{F}(x, y, z) = \Big(xy\ ,\ e^y\ ,\ \arctan(xyz)\Big)$.

Solution: $I = 0$.

Problem 22. Consider the surface $S = \{\,(x, y, z) \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 = 1,\ y \geq 0\,\}$ with normal vector field $\mathbf{n}$ outer to the unit sphere and the function $\mathbf{F}(x, y, z) = (x + z, y + z, 2z)$.

i) Compute $\int_S \mathbf{F} \cdot \mathbf{n}$.

ii) Compute $\int_S \text{rot } \mathbf{F} \cdot \mathbf{n}$.

Solution: $i)$ $8\pi/3$; $ii)$ π.

Problem 23. Compute $\int_S \mathbf{F} \cdot \mathbf{n}$ in the following cases, where $\mathbf{n}$ is the unit outer normal in items $i)$ $iii)$ $iv)$, and the unit upper normal (i.e. the third component is positive) in item $ii)$:

i) $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$ and S is the boundary of the cube $0 \leq x,\ y,\ z \leq 1$.

ii) $\mathbf{F}(x, y, z) = (xy, -x^2, x + z)$ and S is the piece of the plane $2x + 2y + z = 6$ in the first octant.

iii) $\mathbf{F}(x, y, z) = (xz^2, x^2y - z^2, 2xy + y^2z)$ and S is the upper semi-sphere $z = \sqrt{a^2 - x^2 - y^2}$.

iv) $\mathbf{F}(x, y, z) = (2x^2 + \cos yz, 3y^2z^2 + \cos(x^2 + z^2), \exp^{y^2} - 2yz^3)$ and S is the surface of the solid defined by the intersection of the cone $z \geq \sqrt{x^2 + y^2}$ with the sphere $x^2 + y^2 + z^2 \leq 1$.

Solution: $i)$ 3; $ii)$ $27/4$; $iii)$ $2\pi a^5/5$; $iv)$ 0.