Chapter 2.2 Local Extrema. Taylor's Polynomial

Problem 1. Find the critical points and the local extrema of the following functions:

i) $f(x,y) = x^2 + 2y^2 - 4y$,

ii) $g(x,y) = x^2 - xy + y^2 + 2x + 2y - 6$,

iii) $h(x,y) = 3x^2 + 2xy + 2x + y^2 - y + 4$,

iv) $k(x,y) = 8x^3 - 24xy + y^3$.

Solution: $i)$ f has a local minimum at $(0,1)$. Its value is $f(0,1) = -2$; $ii)$ g has a local minimum at $(-2,-2)$. Its value is $g(-2,-2) = -10$; $iii)$ h has local minimum at $(-3/4, 5/4)$. Its value is $h(-3/4, 5/4) = 21/8$; $iv)$ k has at $(0,0)$ a saddle point and at $(2,4)$ a local minimum. Its value is $k(2,4) = -64$.

Problem 2. Find the critical points and the local extrema of the following functions:

i) $f(x,y,z) = y^3 + 2x^2 + y^2 + z^2 + 2yz - 4x - y + 2$,

ii) $g(x,y,z) = -z^3 - 2x^2 - y^2 - z^2 + 2yz - 4x - z - 2$,

iii) $h(x,y,z) = x^3 - 4x^2 - 2y^2 - z^2 - 2xz + 3x + 4y + 1$.

Problem 3. Determine the local extrema of the function $f(x,y) = e^{-x^2+\epsilon y^2}$ for $\varepsilon = 0,\ 1,\ -1$.

Solution: If $\varepsilon = 0$, then f has absolute maxima at the points $(0,y)$, $y \in \mathbb{R}$ and $f(0,y) = 1$; if $\varepsilon = 1$, then the unique critical point of f is at $(0,0)$. It is a saddle point; if $\varepsilon = -1$, then the unique critical point of f is again at $(0,0)$. Now f has an absolute maximum at this point and $f(0,0) = 1$.

Problem 4. Decide if the origin $(0,0)$ is a local or global extremum of the following function:

$$g(x,y) = \begin{cases} xy + xy^3 \sin(x/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Hint: Approach $(0,0)$ along two different lines. Choose the first/second line in such a way that the function has a maximum/minimum at 0 respectively.

Solution: $\nabla g(0,0) = (0,0)$ and

$$\frac{\partial g}{\partial x}(x,y) = \begin{cases} y + y^3 \sin(x/y) + xy^2 \cos(x/y) & \text{if} \quad y \neq 0, \\ 0 & \text{if} \quad y = 0, \end{cases}$$

$$\frac{\partial g}{\partial y}(x,y) = \begin{cases} x + 3xy^2 \sin(x/y) - x^2 y \cos(x/y) & \text{if} \quad y \neq 0, \\ 0 & \text{if} \quad y = 0. \end{cases}$$

Finally,

$$\det Hg(0,0) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0,$$

such that the origin will be a saddle point.

Problem 5. Consider the function $\phi : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\phi(\mathbf{x}) = re^{-r}, \qquad r = \|\mathbf{x}\|.$$

i) Find the local and global extrema of ϕ in the cases $N = 1$ and $N = 2$.

ii) Study the differentiability of ϕ at these points.

Solution: If $N = 1$ we have that $\phi(x) = |x|e^{-|x|}$ i.e. we obtain that at the points $x = 1$ and $x = -1$ there will be local maximum but not global since $\phi(x) \geq \phi(0) = 0$. Thus, $x = 0$ is a local and global minimum.

When $N = 2$ we have that

$$\phi(x,y) = \sqrt{x^2+y^2}e^{-\sqrt{x^2+y^2}}.$$

Therefore, the stationary points will be obtained after solving the equation

$$\frac{1}{\sqrt{x^2+y^2}} - 1 = 0 \Rightarrow x^2 + y^2 = 1,$$

together with the possibility of having either $x =$ or $y = 0$. So that, we find the points $(\pm 1, 0)$, $(0, \pm 1)$ and $(\cos\alpha, \sin\alpha)$ for any $\alpha \in \mathbb{R}$ (points on the circumference of radius 1).

Moreover, the origin $(0,0)$ is also a critical point since

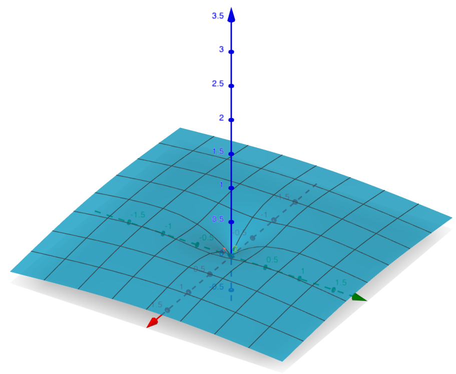
$$\phi(x,y) \geq \phi(0,0) = 0,$$

Actually a global minimum. The points $(\cos\alpha, \sin\alpha)$ are global and local maximum. Also,

$$\lim_{r\to\infty} \phi(r\cos\alpha, r\sin\alpha) = \lim_{r\to\infty} re^{-r} = \lim_{r\to\infty} \frac{r}{e^r} = \lim_{r\to\infty} \frac{1}{e^r} = 0.$$

Indeed, $\phi(\cos\alpha, \sin\alpha) = e^{-1}$, so that

$$0 \leq \phi(x,y) \leq \frac{1}{e}, \quad \forall (x,y) \in \mathbb{R}^2.$$



Problem 6. Classify all critical points of the following functions:

(i) $f(x, y) = \sin(x)\cos(y)$.

(ii) $g(x, y) = \sin(x^2) - \sin(y^2)$.

Solution:

i) (i) Maxima/minima at $\mathbf{x}(k_1, k_2) = (\frac{\pi}{2} + k_1\pi, k_2\pi)$, $k_1, k_2 \in \mathbb{Z}$, if $k_1 + k_2$ is even/odd. Saddle points at $\mathbf{y}(k_1, k_2) = (k_1\pi, \frac{\pi}{2} + k_2\pi)$, $k_1, k_2 \in \mathbb{Z}$.

ii) Saddle point at $(0, 0)$; Minima/saddle points at $\mathbf{x}(k) = (0, \pm\sqrt{\frac{\pi}{2} + k\pi})$, $k \in \mathbb{Z}$, if k is even/odd; Maxima/saddle points at $\mathbf{x}(k) = (\pm\sqrt{\frac{\pi}{2} + k\pi}, 0)$, $k \in \mathbb{Z}$, if k is even/odd; saddle points at $\mathbf{x}(k_1, k_2) = (\pm\sqrt{\frac{\pi}{2} + k_1\pi}, \pm\sqrt{\frac{\pi}{2} + k_2\pi})$, $k_1, k_2 \in \mathbb{Z}$, if $k_1 + k_2$ is even and minimum/maximum at $\mathbf{x}(k_1, k_2) = (\pm\sqrt{\frac{\pi}{2} + k_1\pi}, \pm\sqrt{\frac{\pi}{2} + k_2\pi})$, $k_1, k_2 \in \mathbb{Z}$ if (k_1 is odd,k_2 is even)/(k_1 is even, k_2 is odd).

Problem 7. Write down Taylor's second-order formula for the following scalar fields, close to the origin:

$$i) \quad f(x, y) = \sin(x^2 + y^2), \qquad ii) \quad f(x, y) = e^{x+y},$$

$$iii) \quad f(x, y) = \tan(x + y), \qquad iv) \quad f(x, y) = \sin x \sin y.$$

Solution:

i) $P_{2,(0,0)}(x, y) = 2x^2 + y^2$.

Problem 8. Power-expand the following polynomials in terms of the specified variables:

i) $f(x, y) = x^2 + xy + y$, as powers of $(x - 2)$ and $(y + 1)$.

ii) $f(x, y) = x^2 + y^2 - xy$, as powers of $(x - 1)$ and $(y - 2)$.

iii) $f(x,y) = x^3 + y^2 + xy^2$, as powers of $(x-1)$ and $(y-2)$.

i) $P_{2,(2,-1)}(x,y) = 1 + x - 2 + 3(y+1) + 2(x-2)^2 + 2(y+1)(x-2) + (y+1)^2$.

Problem 9. Let h be a real function of a single real variable, which is differentiable close to -1, and such that $h(-1) = 1$. We define the two-variable function

$$f(x,y) = h(xy) + 2h(y/x) - 4, \quad x \neq 0.$$

i) Find $\nabla f(-1,1)$ in terms of $h'(-1)$.

ii) Write down Taylor's first order polynomial for f around $(-1,1)$.

iii) Compute $h'(-1)$ knowing that the previous polynomial vanishes at $(0,0)$.

Solution:

i) $\nabla f(-1,1) = h'(-1)(-1,-3)$.

ii) $P(x,y) = h'(-1)(2 - x - 3y) - 1$.

iii) $h'(-1) = \frac{1}{2}$.