Local extrema

$f: \mathbb{R}^n \to \mathbb{R}$.

$$\begin{cases} x_0 \text{ is a maximum if } f(x) \leq f(x_0), \text{ for any } x \sim x_0 \\ x_0 \text{ is a minimum if } f(x) \geq f(x_0), \text{ for any } x \sim x_0 \end{cases}$$

stationary point or critical point

$$\nabla f(x_0) = 0$$

Theorem

$\Omega \subset \mathbb{R}^n$, $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ differentiable.

and x_0 local extrema $\Rightarrow$ x_0 is a critical point

$$\nabla f(x_0) = 0$$

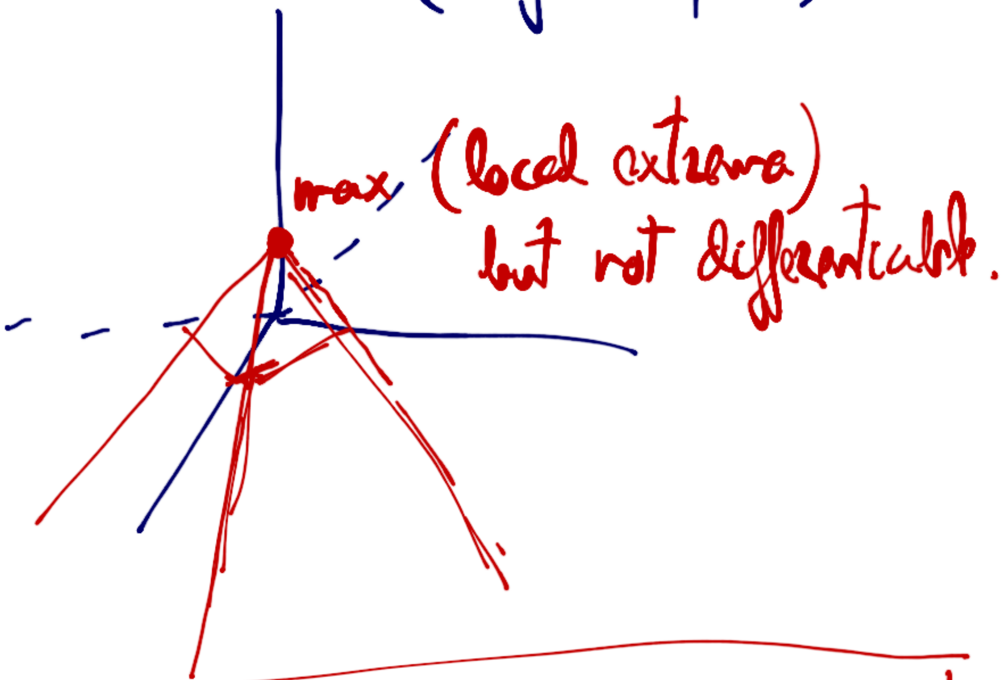
Remark

$\nabla f(x\ \ )=0$ then all

If one of the partial derivatives at x_0 does not exist, f is not differentiable

There is not tangent plane.

but we might have a local extrema
(angular point)



How to find local extrema ?

We might find those local extrema just applying algebraic computations.

Example: $f(x,y) = 3x^2 + y^2 - 6x - 4y + 8$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ differentiable in $\mathbb{R}^2$

Find stationary points $\nabla f = (0,0)$

$$\frac{\partial f(x,y)}{\partial x} = 6x - 6 = 0 \Rightarrow x = 1$$

$$\frac{\partial f(x,y)}{\partial y} = 2y - 4 = 0 \Rightarrow y = 2$$

stationary point $(1,2)$

$\dots + 4 - 6 - 8 + 8$

$= 3(x-1)^2 + (y-2)^2 + 1 \geq 1 \Rightarrow (1,2)$ minimum

Example: Find the critical points for

$$f(x,y) = 1 - \sqrt{x^2+y^2}$$

- We look for point where $\nabla f = (0,0)$
- We look for non-differentiable points.

$$\frac{\partial f(x,y)}{\partial x} = \frac{-x}{\sqrt{x^2+y^2}} \qquad \frac{\partial f(x,y)}{\partial y} = \frac{-y}{\sqrt{x^2+y^2}}$$

A possible local extrema will be $(0,0)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are not defined at $(0,0)$

$$f(0,0) = 1 \quad \Big\} \Rightarrow f(x,y) \leq f(0,0)$$

* Another method to identify critical points

Intersection by vertical planes

$\Downarrow$

existence of saddle points

Example: $f(x,y) = 1 - x^2 + y^2$, $f: \mathbb{R}^2 \to \mathbb{R}$.

Critical points: $\dfrac{\partial f(x,y)}{\partial x} = -2x = 0 \Rightarrow x = 0$

$\dfrac{\partial f(x,y)}{\partial y} = 2y = 0 \Rightarrow y = 0$

One critical point $(0,0)$

$f(0,0) = 1$

In this direction f is bigger

$f(x,y) = 1 - x^2 + y^2 \Big\} \Rightarrow f(0,y) = 1 + y^2 \geq 1$

than 1.

$y = 0$

$(0,0)$

Therefore, $(0,0)$ is a saddle point

- Algebraic computations seem complicated. More systematic method to classify critical points:
  - Quadratic forms
  - Hessian matrix ("second derivatives")

Definition - Quadratic forms [Linear Algebra - Lay]

$Q : \mathbb{R}^N \longrightarrow \mathbb{R}$ quadratic form

$$Q(x) = \sum_{i,j=1}^{N} a_{ij} x_i x_j = x^T A x$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \qquad \boxed{A = (a_{ij})_{i,j=1,\dots,N}}$$

- $Q(x)$ is positive definite $\boxed{\text{if } Q(x) > 0}$ $\forall x \in \mathbb{R}^N$
- $Q(x)$ is negative definite if $\boxed{Q(x) < 0}$ $\forall x \in \mathbb{R}^N$
- Indefinite $\boxed{Q(x) < 0,\ Q(y) > 0 \text{ for some } x, y \in \mathbb{R}^N}$

If A is symmetric, there exists $\{u_1 \dots u_N\}$

D

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$u_i \equiv$ eigenvectors

$\lambda_i \equiv$ eigenvalues.

Then,

$$Q(x) = \underline{x^T A x} = y^T D y = \underbrace{\lambda_1 y_1^2 + \dots + \lambda_N y_N^2}_{\text{spectral theorem.}}$$

Ex: Quadratic form

$$Q(x,y) = \overbrace{3x^2 + y^2} - 6x - 4y + 8$$

$$= (x,y)\, A \begin{pmatrix} x \\ y \end{pmatrix} + 8$$

Hessian matrix

$f: \mathbb{R}^N \longrightarrow \mathbb{R}$ scalar function

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_N \partial x_1} \\ \vdots & & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_N} & \frac{\partial^2 f}{\partial x_2 \partial x_N} & \cdots & \frac{\partial^2 f}{\partial x_N^2} \end{pmatrix}$$

If $f \in C^2$ (all second derivatives are continuous) the Hessian matrix is symmetric (Schwarz's Th)

Remark

Hessian criteria in $\mathbb{R}^2$

$$Hf(x,y) = \begin{pmatrix} \dfrac{\partial^2 f(x,y)}{\partial x^2} & \dfrac{\partial^2 f(x,y)}{\partial y\, \partial x} \\ \dfrac{\partial^2 f(x,y)}{\partial x\, \partial y} & \dfrac{\partial^2 f(x,y)}{\partial y^2} \end{pmatrix} \quad f \in C^2$$

$f : \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathrm{Dom} f$ with

$$\boxed{\frac{\partial f(x_0 y_0)}{\partial x} = 0 = \frac{\partial f(x_0 y_0)}{\partial y}}$$

a) If $\det Hf(x_0, y_0) > 0$ and $\boxed{\dfrac{\partial^2 f(x_0 y_0)}{\partial x^2} > 0}$

then $(x_0 y_0)$ is a local minimum.

$$\det Hf(x_0, y_0) = \frac{\partial^2 f(x_0 y_0)}{\partial x^2} \cdot \frac{\partial^2 f(x_0 y_0)}{\partial y^2} - \left(\frac{\partial^2 f(x_0 y_0)}{\partial x\, \partial y} \right)^2 > 0$$

So $\dfrac{\partial^2 f(x_0,y_0)}{\partial x^2} > 0$ and $\dfrac{\partial^2 f(x_0,y_0)}{\partial y^2} > 0$

f is concave upwards in both directions.

$\boxed{b)}$ $\det Hf(x_0,y_0) > 0$ and $\dfrac{\partial^2 f(x_0,y_0)}{\partial x^2} < 0$

$(x_0\,y_0)$ local maximum

$\boxed{c)}$ $\det Hf(x_0 y_0) < 0$ saddle point

d) $\det Hf(x_0,y_0) = 0$ not conclusive !!

degenerate critical point.

Example: $f(x,y) = x^3 - 6xy + 3y^2 - 1$

First find critical points

$$\left.\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= 3x^2 - 6y = 0 \\ \frac{\partial f(x,y)}{\partial y} &= -6x + 6y = 0 \end{aligned}\right\} \Rightarrow x = y$$

so that $3x^2 - 6x = 0 \Rightarrow x(3x-6) = 0$

$$x = 0, \quad x = 2$$

Two critical points $(0,0), (2,2)$

Hessian matrix

$$\det Hf(0,0) = \det \begin{vmatrix} 0 & -6 \\ -6 & 6 \end{vmatrix} = -36 < 0$$

This means $(0,0)$ is a saddle point

$$\det Hf(2,2) = \det \begin{vmatrix} 12 & -6 \\ -6 & 6 \end{vmatrix} = 72 - 36 = 36 > 0$$

$$\frac{\partial^2 f(2,2)}{\partial x^2} = 12 > 0$$

$(2,2)$ is a local minimum

Theorem – Taylor's polynomial (of second order)

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ with continuous partial derivatives

$f \in C^2$. Then, $(x_0 y_0) \in \mathrm{Dom}\, f$.

$$f(x,y) = \underbrace{f(x_0 y_0) + \frac{\partial f(x_0 y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0 y_0)}{\partial y}(y-y_0)}_{\text{linear part.}}$$

$$\underbrace{+ \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 f(x_0 y_0)}{\partial x_i \, \partial x_j}(x-x_0)(y-y_0)}_{\text{quadratic form.}} + R$$

$$\frac{R}{\|(x,y)-(x_0,y_0)\|^2} \longrightarrow 0$$

Example: $f(x,y) = e^x \cos y$

Taylor's polynomial around $(x_0, y_0) = (0,0)$

$$f(0,0) = 1$$

$$\frac{\partial f(x,y)}{\partial x} = e^x \cos y \Rightarrow \frac{\partial f(0,0)}{\partial x} = 1$$

$$\frac{\partial f(x,y)}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial f(0,0)}{\partial y} = 0$$

$$\nabla f(0,0) = (1,0)$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = e^x \cos y \Rightarrow \frac{\partial^2 f(0,0)}{\partial x^2} = 1$$

$$Hf(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = -e^x \sin y = \frac{\partial^2 f(x,y)}{\ldots}$$

Then,

$$f(x,y) = 1 + (1,0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\underbrace{(x\ y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} + R$$

$$= 1 + x + \frac{1}{2}(x, -y)\begin{pmatrix} x \\ y \end{pmatrix} + R$$

$$= \underbrace{1 + x + \frac{(x^2 - y^2)}{2}}_{\text{Taylor's polynomial of second order for } f(x,y) \text{ around } (0,0)} + R$$

Taylor's polynomial of second order for $f(x,y)$ around $(0,0)$

$$\frac{\partial^2 f}{\quad} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Computations of local extrema

Assume $f: \mathbb{R}^N \to \mathbb{R}$ is of class C^2

a) Local extrema belong to the set of critical points

b) The type of local extrema is given by a quadratic form.

$$\underline{Q_H(x_1, \ldots x_N)} = x^T \underbrace{Hf}_{\text{Hessian matrix.}} x = y^T D y = \underline{\lambda_1 y_1^2 + \cdots + \lambda_N y_N^2}$$

$\{y_1, \ldots y_N\}$ is an orthogonal basis.

Then, a) If all eigenvalues λ_i are positive $\Rightarrow Q_H$ positive

c) If we have positive and negative eigenvalues then Q_H indefinite. $Q_H = \lambda_1 y_1^2 + \dots + \lambda_N y_N^2$

The idea is to apply the quadratic form to get local extrema using Taylor's polynomial.

For example in $\mathbb{R}^2$

$$f(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \frac{1}{2}(x-x_0 \; y-y_0)\, Hf(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

If (x_0,y_0) is a critical point $\nabla f(x_0,y_0)=0$

quadratic form.

So

Then $f(x,y) > f(x_0,y_0)$

(x_0,y_0) is a local minimum.

Writing the quadratic form

$$\underline{Q_H(x,y)} = \underline{\lambda_1 y_1^2 + \lambda_2 y_2^2} \underline{> 0}$$

$$\Downarrow \qquad \Uparrow$$

$$\lambda_1, \lambda_2 > 0$$

This is equivalent to

$$\boxed{\det\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} > 0} \quad \text{and } \lambda_1 > 0, \ \lambda_2 > 0$$

If $Q_H < 0$

$f(x\,y) < f(x_0\,y_0) \Rightarrow (x_0\,y_0)$ local max.