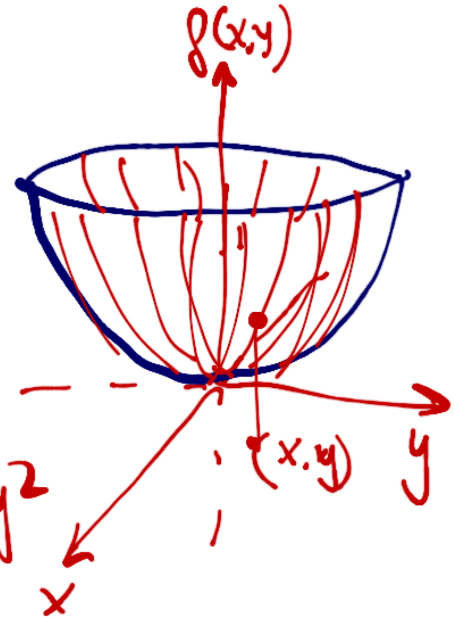
Summary Differentiability in $\mathbb{R}^N$

A function $f: \mathbb{R}^2 \to \mathbb{R}$ assigns any arbitrary point in $\mathbb{R}^2$ to just one value in $\mathbb{R}$

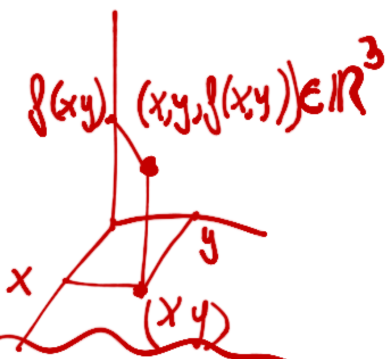
$$(x,y) \longrightarrow f(x,y) \in \mathbb{R}$$



Example: $f(x,y) = x^2+y^2$

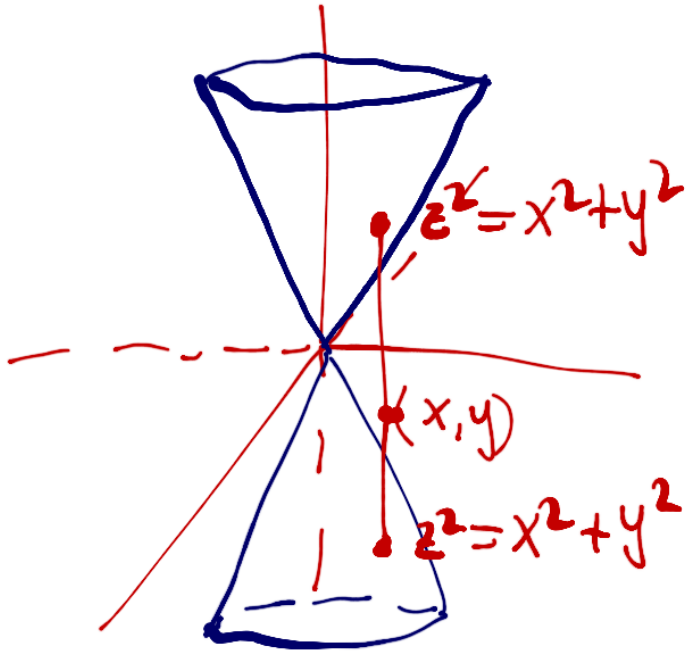
$$(x,y) \longrightarrow f(x,y) = x^2+y^2$$

Its graph is a surface!!



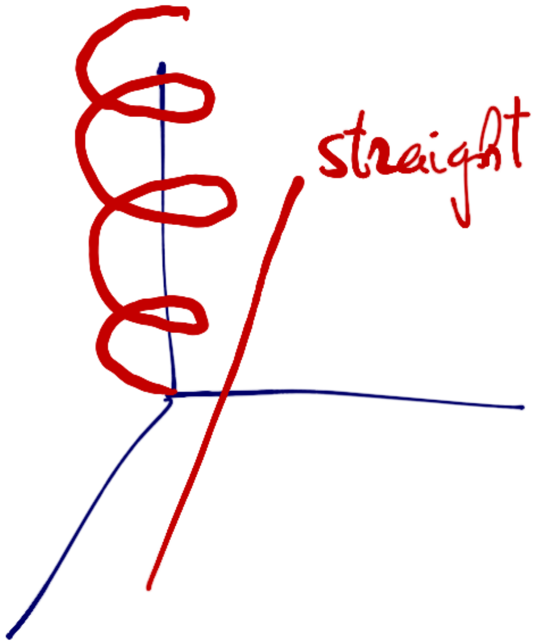
* Such a surface can be written

$z^2 = x^2 + y^2$ Cone



Different functions:

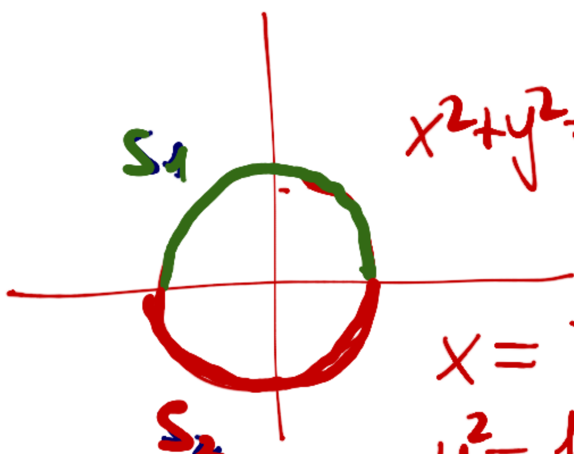
- Scalar functions: $f: \mathbb{R}^N \to \mathbb{R}$ (magnitudes)
- Vector functions: $F: \mathbb{R}^N \to \mathbb{R}^M$



straight line.

$s(t) \equiv$ parametric expression.

$s(t) \in \mathbb{R}$ given by a parameter t.



$x^2+y^2=1$

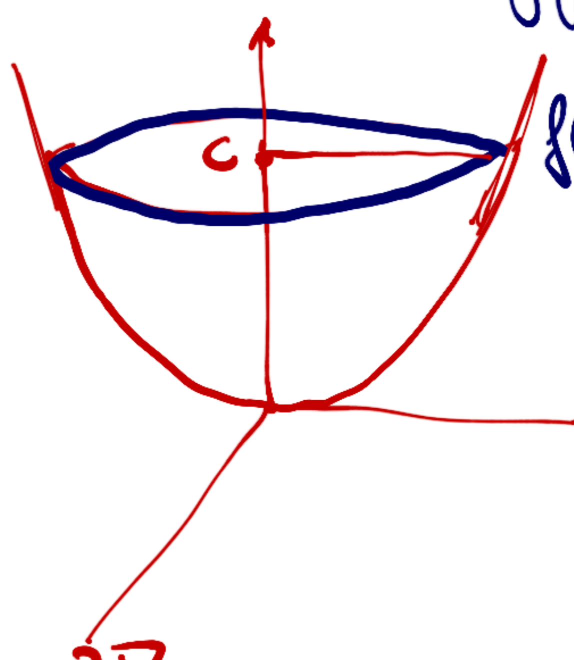
$$\left.\begin{array}{l} x = t \\ y^2 = 1-t^2 \end{array}\right\}$$

Level curves (or level surfaces)

Curves where the function is constant

$$f(x,y) = c \ , \ c \in \mathbb{R} .$$

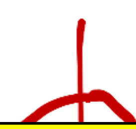
They allow us to draw 3D figures into 2D.



$f(x,y)=c$

$f(x,y) = x^2+y^2$

Level curve : $x^2+y^2=c$ in 2D

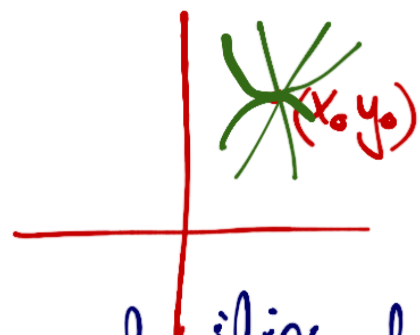


Limits and continuity

$\boxed{\text{If the limit exists it must be unique}}$

This means it **CANNOT DEPEND** on the trajectory.

In 2D



- Approaching following families of functions $\left.\begin{array}{l} y = \lambda x \\ y = \lambda x^2 \end{array}\right\}$ around the point

around $(0,0)$

DOES NOT IMPLY

depend on θ.

<u>Continuity at x_0</u>: $\lim_{x \to x_0} f(x) = f(x_0) \in \mathbb{R}$.

$x_0 \in \mathbb{R}^N$

<u>Differentiability at x_0</u>:

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

direction.

Derivatives: Partial derivatives.

definition
$$\begin{cases} \dfrac{\partial f(x_0 y_0)}{\partial x} = \lim_{h \to 0} \dfrac{f((x_0, y_0) + h(1,0)) - f(x_0, y_0)}{h} \\[2ex] \dfrac{\partial f(x_0 y_0)}{\partial y} = \lim_{k \to 0} \dfrac{f((x_0 y_0) + k(0,1)) - f(x_0 y_0)}{k} \end{cases}$$

direction of Y

ratio of change in the direction of the axis.
(vector of standard basis.)

values for derivatives.

Directional derivative. $D_v f$
ratio of change in the direction v

* Existence of derivatives
  - DOES NOT IMPLY diff..
  - DOES NOT IMPLY Cont.

* Differentiability means existence of tangent plane (smooth function)

Analytically: Existence of partial derivatives

+

$$0 = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\|(x,y)-(x_0,y_0)\|}$$

$f: \mathbb{R}^2 \to \mathbb{R}$.

or

Existence + continuity of partial derivatives

$\Downarrow$

differentiability

$$\lim \frac{\left\| F(x,y) - F(x_0,y_0) - JF(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \right\|}{} = 0$$

In $\mathbb{R}^3$, $f: \mathbb{R}^3 \to \mathbb{R}$

$$\frac{f(x,y,z) - f(x_0,y_0,z_0) - \frac{\partial f}{\partial x}(x-x_0) - \frac{\partial f}{\partial y}(y-y_0) - \frac{\partial f}{\partial z}(z-z_0)}{\|(x,y,z)-(x_0,y_0,z_0)\|}$$

$$Df = \nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

If f is diff. / f a function

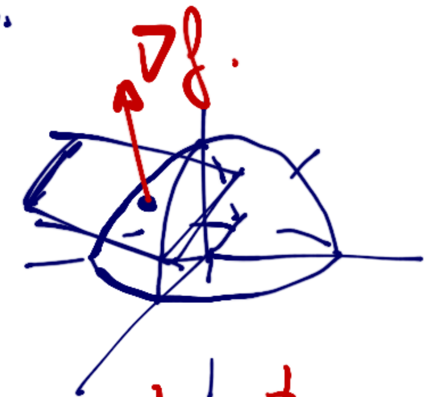
$$D_v f(x_0) = \langle \underbrace{\nabla f(x_0)}_{\text{gradient}}, v \rangle \quad \text{with } \|v\|=1$$

$$+ \frac{\partial f}{\partial y}(y-y_0)$$

- Tangent plane to a surface.

$$\boxed{f(x,y,z)=0}$$

Implicit form.



gradient vector is orthogonal to the surface

Tangent plane:

$$\underbrace{\nabla f(x_0,y_0,z_0)\cdot\begin{pmatrix}x-x_0\\ y-y_0\\ z-z_0\end{pmatrix}=0}$$

Example: $z^2=x^2+y^2$ Cone ~ $f(x,y,z)=x^2+y^2-z^2 = 0$

Tangent plane at $(1,1,1)$

$(2x, 2y, -2z)$

$(z-1)$

• $f(x,y) = x^2+y^2$ function.

Equivalent to $\underbrace{z = x^2+y^2}_{\text{Formula.}} \equiv$ surface.

$$g(x,y,z) = x^2+y^2-z.$$

$$\nabla g(x,y,z) = (2x, 2y, -1)$$

At $(1,1,2)$ on the surface.

$$f(1,1)=2 \qquad \nabla f = (2x,2y) \Rightarrow \nabla f(1,1) = (2,2)$$

option a): $z = f(1,1) + \nabla f(1,1) \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$

$$z = 2 + 2(x-1) + 2(y-1)$$

$(z-2)$ $(z-2)$

Chain rule:

$$D(g\circ f)(x_0) = Dg(f(x_0)) \cdot Df(x_0)$$

matrices.

- In 1D $y=f(x)$ and $x=x(t)$

then $y=f(x(t)) = h(t)$

So $$\frac{dy}{dt} = \underbrace{\frac{dy}{dx}}_{\frac{df(x(t))}{dt}} \frac{dx}{dt} = \frac{df(x(t))}{dx} \cdot \frac{dx}{dt}$$

In 2D $\quad \underline{\underline{z = f(x,y)}}, \quad x \equiv x(t)$

$$y \equiv y(t)$$

$\underline{\underline{f : \mathbb{R}^2 \to \mathbb{R}}}. \quad Df = \nabla f$

$$z = f\big(x(t), y(t)\big) = f\big(S(t)\big)$$

$$S(t) = \big(x(t), y(t)\big)$$

$$S : \mathbb{R} \to \mathbb{R}^2$$

Thanks to the chain rule

$$\frac{dz}{dt} = \frac{df(S(t))}{dS} \cdot \frac{dS}{dt} = \nabla f\big(S(t)\big) \cdot \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

$$\left(\partial z \quad \partial z \right) \begin{pmatrix} \frac{dx}{dt} \\ \end{pmatrix} = \partial z \; \partial x$$

Example: $\boxed{z = x^2 y + 3xy^4}$

$x = \sin 2t$

$y = \cos t$

Find $\dfrac{dz}{dt}$

$$z = x^2(t)\, y(t) + 3\, x(t)\, y^4(t)$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

$$= \underbrace{\left(2xy + 3y^4\right)}_{\frac{\partial z}{\partial x}} \underbrace{2\cos 2t}_{\frac{dx}{dt}} - \underbrace{\left(x^2 + 12xy^3\right)}_{\frac{\partial z}{\partial y}} \underbrace{\sin t}_{\frac{\partial y}{\partial t}}$$

Problem 10 in $\boxed{1.3}$

ii) $s'(t) = F(s(t))$

$s(t) = \left(e^{2t}, \frac{1}{t}, \log t\right)$, $F(x,y,z) = (2x, -y^2, y)$

$\Downarrow$

$$s'(t) = \frac{ds(t)}{dt} = \left(2e^{2t}, -\frac{1}{t^2}, \frac{1}{t}\right)$$

$$F(s(t)) = F\left(e^{2t}, \frac{1}{t}, \log t\right) =$$

$$= \left(2e^{2t}, -\frac{1}{t^2}, \frac{1}{t}\right)$$

• The pressure P (in kilopascals) volume V (liters) and temperature T (in Kelvins) of a mole of an ideal gas are related by the equation

$$\boxed{PV = 8'31\,T}$$

Find the rate at which pressure is changing when $T = 300K$ and increasing at a rate of $0'1\ K/s$ and $V = 100L$ and increasing at a rate of $0'2\ L/s$

rate of change for $P \equiv \frac{dP}{dt}$, $P = \frac{8'31\,T}{V}$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\underbrace{\frac{dT}{dt}}_{0'1} + \frac{\partial P}{\partial V}\underbrace{\frac{dV}{dt}}_{0'2}$$

$$= \frac{8'31}{V}\cdot 0'1 - \frac{8'31\,T}{V^2}\cdot 0'2$$

$$= \frac{8'31}{100}\cdot 0'1 - \frac{8'31\cdot 300}{(100)^2}\cdot 0'2 = \underline{-0'04155}$$

The pressure decreases at that rate.

Implicit differentiation

Write a function

Implicit form	Explicit form	Derivative
$xy=1$	$y=\frac{1}{x} \Rightarrow$	$\frac{dy}{dx}=-\frac{1}{x^2}$

However we cannot use those two forms when we are unable to solve the equation for y as a function of x

$$\boxed{x^2-2y^3+4y=2}$$

- Every term depending on x we differentiate as usual
- Every term involving $y \equiv y(x)$ we must apply the chain rule.

Example: $x^2+y^2=1$, $y \equiv y(x)$

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[1]=0$$

$$2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy(x)}{dx}=\frac{-x}{y}$$

In several variables.

$$\boxed{\frac{dy}{dx} = - \frac{\partial F/\partial x}{\partial F/\partial y}}$$

Ex: $x^3 + y^3 = 3xy \Rightarrow F(x,y) = x^3 + y^3 - 3xy$

$$\frac{dy}{dx} = - \frac{\partial F/\partial x}{\partial F/\partial y} = - \frac{3x^2 - 3y}{3y^2 - 3x} = - \frac{x^2 - y}{y^2 - x}$$

explict form

If $\boxed{z = f(x,y)}$ in implicit form $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$F(x, y, f(x,y)) = 0 \quad (x,y)$ indept variables.

Using the chain rule.

Infectious disease outbreak size.

Mathematical models predict the fraction of population that will be infected when a disease begins to spread.

$$\rho e^{-q\Delta} - 1 + \Delta = 0$$
$$F(\rho, q, \Delta)$$

$$\boxed{\rho e^{-q\Delta} = 1 - \Delta}$$

Kermack-McKendrick model

$\Delta \equiv$ fraction of population infected.

$\rho \equiv$ initially population susceptible to infection

$q \equiv$ measure of transmissibility

$$\boxed{\frac{\partial \Delta}{\partial q} = -\frac{\partial F/\partial q}{\partial F/\partial \Delta} = \frac{\square}{\square}}$$

$$F(\ell, q, \Delta(q)) = 0$$

$$\underbrace{\frac{\partial F}{\partial \ell}\cdot\underbrace{\frac{d\ell}{dq}}_{=0}}_{=0} + \underbrace{\frac{\partial F}{\partial q}\underbrace{\frac{dq}{dq}}_{=1}} + \frac{\partial F}{\partial \Delta}\frac{\partial \Delta(q)}{\partial q} = 0$$

$$\frac{\partial \Delta}{\partial q} = -\frac{\partial F/\partial q}{\partial F/\partial \Delta}$$