Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 8

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Outline

1. Properties; Higher-order Partial Derivatives
   - Properties of Differentiation
   - kth order derivatives and Schwarz Theorem

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Differentiation is a linear operation:

Proposition 4.1: Linearity of Differentiation

- Let $\mathbf{f}, \mathbf{g} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be two functions that are both differentiable at a point $\mathbf{a} \in X$

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  2. The function $\mathbf{k} = c\mathbf{f}$ is differentiable at $\mathbf{a}$, and
  $$D\mathbf{k}(\mathbf{a}) = D(c\mathbf{f})(a) = cD\mathbf{f}(\mathbf{a})$$

Example 1

- Let **f** and **g** be defined by,

$$\begin{aligned}\mathbf{f}(x,y) &= (x+y, xy\sin y, y/x)\\ \mathbf{g}(x,y) &= (x^2+y^2, ye^{xy}, 2x^3-7y^5)\end{aligned}$$

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- $\mathbf{f}$ is differentiable only in $\mathbb{R}^2 \setminus \{x=0\}$ and $\mathbf{g}$ is differentiable on all of $\mathbb{R}^2$.

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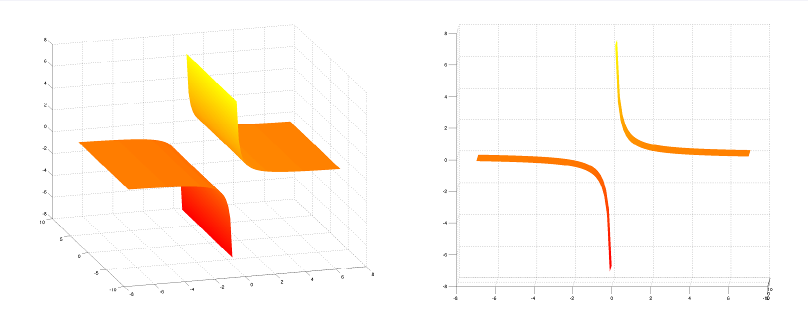
- If we let $\mathbf{h}=\mathbf{f}+\mathbf{g}$, then Proposition 4.1 tells us that $\mathbf{h}$ must be differentiable on all of its domain
- Furthermore,

$$\begin{aligned}D\mathbf{h}(x,y) &= D\mathbf{f}(x,y)+D\mathbf{g}(x,y)\\ &= \begin{bmatrix} 2x+1 & 2y+1\\ y\sin y+y^2e^{xy} & x\sin y+xy\cos y+e^{xy}+xye^{xy}\\ 6x^2-y/x^2 & 1/x-35y^4\end{bmatrix}\end{aligned}$$

Example 1

- Some graphical representation.

$$\frac{\partial f_3}{\partial y} = 1/x$$



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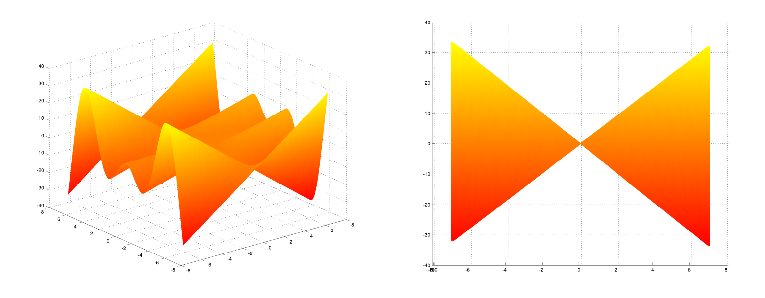

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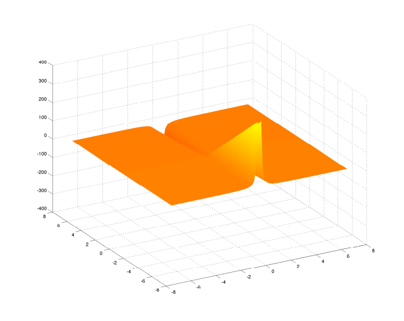

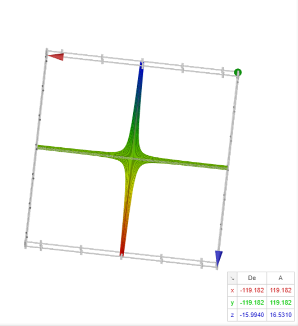

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  2. If $g(\mathbf{a}) \neq 0$, then the quotient function f/g is differentiable at $\mathbf{a}$:
  $$D(f/g)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$$

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- So that

$$D(fg)(x,y,z) = \begin{bmatrix} (yz - z^2)e^{xy} + (xyz + 2yz^2 - xz^2)ye^{xy} \\ (xz + 2z^2)e^{xy} + (xyz + 2yz^2 - xz^2)xe^{xy} \\ (xy + 4yz - 2xz)e^{xy} \end{bmatrix}^T$$

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$$\begin{aligned} f(x,y,z) &= ze^{xy} \\ g(x,y,z) &= xy + 2yz - xz \\ Df(x,y,z) &= \begin{bmatrix} yze^{xy} & xze^{xy} & e^{xy} \end{bmatrix} \\ Dg(x,y,z) &= \begin{bmatrix} y-z & x+2z & 2y-x \end{bmatrix} \end{aligned}$$

- Using Proposition 4.2

$$g(x,y,z)Df(x,y,z) + f(x,y,z)Dg(x,y,z) =$$

$$= \begin{bmatrix} (xy^2z + 2y^2z^2 - xyz^2)e^{xy} \\ (x^2yz + 2xyz^2 - x^2z^2)e^{xy} \\ (xy + 2yz - xz)e^{xy} \end{bmatrix}^T + \begin{bmatrix} (yz - z^2)e^{xy} \\ (xz + 2z^2)e^{xy} \\ (2yz - xz)e^{xy} \end{bmatrix}^T$$

$$= e^{xy} \begin{bmatrix} (yz - z^2) + (xyz + 2yz^2 - xz^2)y \\ (xz + 2z^2) + (xyz + 2yz^2 - xz^2)x \\ (xy + 4yz - 2xz) \end{bmatrix}^T$$

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$$\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_2} \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial}{\partial x_{i_2}} \frac{\partial}{\partial x_{i_1}} (f(x_1, x_2, \ldots, x_n))$$

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- Equivalent notation,

$$f_{x_{i_1} x_{i_2} \cdots x_{i_k}}(x_1, x_2, \ldots, x_n)$$

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$$\begin{aligned} f_{yw}(x,y,z,w) &= \frac{\partial^2 f}{\partial w \partial y} = \frac{\partial}{\partial w}\frac{\partial}{\partial y}(xyz + xy^2w - \cos(x+zw)) \\ &= \frac{\partial}{\partial w}(xz + 2xyw) = 2xy \\ f_{wy}(x,y,z,w) &= \frac{\partial^2 f}{\partial w \partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial w}(xyz + xy^2w - \cos(x+zw)) \\ &= \frac{\partial}{\partial y}(xy^2 + z\sin(x+zw)) = 2xy \end{aligned}$$

This example suggests that there might be a simple relationship among the mixed second partials

Theorem 4.3 (Schwarz)

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- That is, if i_1 and i_2 are any two integers between 1 and n, then,

$$\frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}} = \frac{\partial^2 f}{\partial x_{i_2} \partial x_{i_1}}$$

Definition 4.4: Smooth Functions

- Assume X is open in $\mathbb{R}^n$.
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A vector-valued function $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is of class $C^k(C^\infty)$

if and only if

Each of its component functions is of class $C^k(C^\infty)$

Theorem 4.5 Schwarz (extended)

- Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of class C^k
- Then the order in which we calculate any kth-order partial derivative does not matter
- Suppose
  - $(i_1, \ldots, i_k)$ are any k integers (not necessarily distinct) between 1 and n, and
  - $(j_1, \ldots, j_k)$ is any permutation (rearrangement) of these integers
- Then

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} = \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}$$

Example 5

- Let

$$f(x,y,z,w) = x^2 w e^{yz} - z e^{xw} + xyzw$$

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- We verify Theorem 4.5
$$\frac{\partial^5 f}{\partial x \partial w \partial z \partial y \partial x} = 2e^{yz}(yz + 1) = \frac{\partial^5 f}{\partial z \partial y \partial w \partial^2 x}$$