Problem 3.2.5 Use the Taylor expansion of the function arctg x to study the convergence of the series

$$\sum_{n=1}^{\infty}\left(\operatorname{arctg}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n}}\right).$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad -1 < x < 1$$

Luego: $\arctan\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{1}{3}\left(\frac{1}{\sqrt{n}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{n}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{n}}\right)^7 + \cdots\cdots$

$$\operatorname{arctg}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n}} = -\frac{1}{3}\left(\frac{1}{\sqrt{n}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{n}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{n}}\right)^7 + \cdots\cdots$$

$$\sum_{n=1}^{\infty}\left(\operatorname{arctg}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n}}\right) = \sum_{n=1}^{\infty}\left(-\frac{1}{3}\left(\frac{1}{\sqrt{n}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{n}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{n}}\right)^7 + \cdots\right)$$

$$= -\sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{3}\left(\frac{1}{\sqrt{n}}\right)^3 + \frac{1}{7}\left(\frac{1}{\sqrt{n}}\right)^7 + \cdots\right)}_{a_n} + \sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{5}\left(\frac{1}{\sqrt{n}}\right)^5 + \frac{1}{9}\left(\frac{1}{\sqrt{n}}\right)^9\cdots\right)}_{c_n}$$

Sean las series:

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \rightsquigarrow b_n = \frac{1}{n^{3/2}} \qquad P = \frac{3}{2} \rightsquigarrow \text{convergente}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{5/2}} \rightsquigarrow d_n = \frac{1}{n^{5/2}} \qquad P = \frac{5}{2} \rightsquigarrow \text{convergente}$$

Por criterio de comparación

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{3}\left(\frac{1}{\sqrt{n}}\right)^3 + \frac{1}{7}\left(\frac{1}{\sqrt{n}}\right)^7 + \cdots}{\frac{1}{n^{3/2}}} = \frac{1}{3} \rightsquigarrow \sum_{n=1}^{\infty} a_n \text{ converge}$$

$$\lim_{n\to\infty}\frac{c_n}{d_n} = \lim_{n\to\infty}\frac{\frac{1}{5}\left(\frac{1}{\sqrt{n}}\right)^5 + \frac{1}{9}\left(\frac{1}{\sqrt{n}}\right)^9\cdots)}{\frac{1}{n^{5/2}}} = \frac{1}{5} \rightsquigarrow \sum_{n=1}^{\infty} c_n \text{ converge}$$

Como a_n y b_n convergen $\Rightarrow \displaystyle\sum_{n=1}^{\infty}\left(\operatorname{arctg}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n}}\right)$ converge